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Proving The Fermat Last Theorem For Case $q \leq n$

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A B S T R A C T

Fermat's Last Theorem is a well-known classical theorem in mathematics. Andrew Willes has proven this theorem using the modular elliptic curve. However, the proposed proof is difficult for mathematicians and researchers to understand. For this reason, in this study, we provide evidence of several properties of Fermat's Last Theorem with a simple concept. We use Newton's Binomial Theorem, well-known in Fermat's time. In this study, we prove Fermat's Last Theorem for case $q \leq n$.

Keywords: Fermat's last theorem, Newton Binomial, proof of Fermat

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1. Introduction

Andrew Willes has proved the Fermat Last Theorem with the modular elliptic curve [1]. Willes' proof is quite complex to understand and may not have been known in Fermat's time. After Willes' proof, other researchers have also provided an alternative to the Fermat Last Theorem through several new Diophantine and Hilbert–Waring hybrid equations [2]. In [3], Fermat's Last Theorem is proved by using Reduction ad absurdum, the Pythagorean Theorem, and congruent triangles' properties.

In addition, other researchers have also tried to provide proof of the Fermat Last Theorem in an easy way [4]–[8]. The researchers proved the Fermat Last Theorem through unique factorization [9], contra positive method [10], least (smallest) deviation [11], group theoretical & calculus [12], and elliptic curves methods[13].

For this reason, in this study, we examine some of the properties of the proof of Fermat's Last Theorem using Newton's Binomial Theorem. We try to provide a more straightforward proof of Fermat's last theorem that may have existed in Fermat's time.

2. Preliminaries

In this research, we need the concept of combination to build Newton's Binomial Theorem. We also need a positive real number ($\mathbb{R}^+$) and a positive integer ($\mathbb{Z}^+$). Newton's binomial concept is the main tool for proving the fermat last theorem for case $q \leq n$.

Definition 2.1. For any $n, r \in \mathbb{Z}^+$and $r \leq n$, the combination$r$of $n$elements is $C_r^n = \frac{n!}{(n-r)!r!}$.

Theorem 2.2. For any $p, q \in \mathbb{R}^+$ and $n \in \mathbb{Z}^+$, apply $(p+q)^n = p^n + C_1^n p^{n-1} q + \cdots + C_{n-1}^n p q^{n-1} + p^n$[14], [15].

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Furthermore, to guarantee the existence of a root of a positive integer ($\mathbb{Z}^+$), we are given the following theorem.

Theorem 2.3. If $z, n \in \mathbb{Z}^+$, then $\sqrt[n]{z} \in \mathbb{Z}^+$or $\sqrt[n]{z} \in irrational$.

We also need the concept of the greatest common divisor of two nonzero integers.

Definition 2.4. Given $a, b, c \in \mathbb{Z}$ and $a, b \neq 0$. If $(a, b) = c$, then $c|a$ and $c|b$.

3. Main Result

Before we partially prove Fermat's Last Theorem, we'll build some supporting theorems.

Theorem 3.1. Given any $p, q, n \in \mathbb{Z}^+$ and $n > 1$. If $p \geq q$ then applies $p^n < (p^n + q^n) < \left(p + \frac{q}{n}\right)^n$.

Proof. First, we prove $p, q, n \in \mathbb{Z}^+$. Because $p \geq q$ and $p, q, n \in \mathbb{Z}^+$ then $p^n < (p^n + q^n)$ applies.

Second, we prove$(p^n + q^n) < \left(p + \frac{q}{n}\right)^n$. According to Theorem 2.2., we get.

$$\left(p + \frac{q}{n}\right)^n = C_0^n p^n \left(\frac{q}{n}\right)^0 + C_1^n p^{n-1} \left(\frac{q}{n}\right) + \cdots ...$$
$$+ C_{n-1}^n p \left(\frac{q}{n}\right)^{n-1} + C_n^n p^0 \left(\frac{q}{n}\right)^n$$
$$= p^n + np^{n-1}\left(\frac{q}{n}\right) + \cdots ...$$
$$+ np\left(\frac{q}{n}\right)^{n-1} + \left(\frac{q}{n}\right)^n$$
$$= p^n + p^{n-1}q + \cdots ... + np\left(\frac{q}{n}\right)^{n-1} + \left(\frac{q}{n}\right)^n$$

Since $p, q, n \in \mathbb{Z}^+, n > 1$ and $p \geq q$ then apply

$$q \leq p$$
$$\Leftrightarrow q^{n-1} \leq p^{n-1}$$
$$\Leftrightarrow q^{n-1}q \leq p^{n-1}q$$
$$\Leftrightarrow q^n \leq p^{n-1}q$$

Look back at $\left(p + \frac{q}{n}\right)^n = p^n + p^{n-1}q + \cdots .... + np\left(\frac{q}{n}\right)^{n-1} + \left(\frac{q}{n}\right)^n$. Because $q^n \leq p^{n-1}q$ then applies

$$\left(p + \frac{q}{n}\right)^n = p^n + p^{n-1}q + \cdots ... + np\left(\frac{q}{n}\right)^{n-1}$$
$$+ \left(\frac{q}{n}\right)^n \geq p^n + q^n + \cdots ... + np\left(\frac{q}{n}\right)^{n-1} + \left(\frac{q}{n}\right)^n$$

Thus, we get $(p^n + q^n) < \left(p + \frac{q}{n}\right)^n$.

So, for $p, q, n \in \mathbb{Z}^+$, $n > 1$, and $p \geq q$, then $p^n < (p^n + q^n) < \left(p + \frac{q}{n}\right)^n$.

Next, we will try to prove Fermat's Last Theorem partially. In the equation $p^n + q^n = z^n$, we will divide the two conditions, namely $q \leq n$ and $q > n$. The following is a proof of Fermat's Last Theorem for $q \leq n$.

Theorem 3.2. Given any $p, q, n \in \mathbb{Z}^+$and $n > 2$. If $p \geq q$ and $q \leq n$, the equation $p^n + q^n = z^n$ has no integer solution, i.e., $z \notin \mathbb{Z}^+$.

Proof. We take any $p, q, n \in \mathbb{Z}^+$ and $n > 2$, such that there is $z \in \mathbb{R}^+$ that satisfies $p^n + q^n = z^n$. Based on Theorem 3.1., we get

$$p^n < (p^n + q^n) < \left(p + \frac{q}{n}\right)^n$$
$$\Leftrightarrow p^n < z^n < \left(p + \frac{q}{n}\right)^n$$
$$\Leftrightarrow p < z < \left(p + \frac{q}{n}\right)$$

Since $q \leq n$ then $p < z < (p + 1)$, so $z \notin \mathbb{Z}^+$ applies. So, no integer is the solution.

The following example will explain in more detail the explanation of Theorem 3.2.

Example 3.3. We have the equation $200^{1200} + 41^{1200} = z^{1200}$. Based on Theorem 3.2., we get

$$200 < z < 200 + \left(\frac{41}{1200}\right).$$

It is clear that $z \notin \mathbb{Z}^+$.

Example 3.4. We have the equation $1234567^{98766} + 98765^{98766} = z^{98766}$. Based on Theorem 3.2., we get

$$1234567 < z < \left(1234567 + \frac{98765}{98766}\right).$$

It is clear that $z \notin \mathbb{Z}^+$.

Next, we have to prove condition $q > n$. In this study, we have not found condition $q > n$ in general. We found the following properties.

Theorem 3.5. Given any $p, q, n \in \mathbb{Z}^+$and $2 < n < q \leq p$. If $(p, q) = c$ and $\frac{q}{c} \leq n$, the equation $p^n + q^n = z^n$ has no integer solution, i.e., $z \notin \mathbb{Z}^+$.

Proof. Based on Definition 2.4., if $(p, q) = c$, then $c|p$ and $c|q$. Based on the principle of division, there are $k, l \in \mathbb{Z}^+$ so that $p = kc$ and $q = lc$. Note that applies

$$p^n + q^n = z^n$$
$$\Leftrightarrow (kc)^n + (lc)^n = (sc)^n$$
$$\Leftrightarrow k^n + l^n = s^n$$

Based on Theorem 3.2. applies $k < s < \left(k + \frac{l}{n}\right)$. Because of $\frac{q}{c} = \frac{lc}{c} \leq n$, then $l \leq n$. We get $k < s < (k+1)$ so that $s \notin \mathbb{Z}^+$. Based on Theorem 2.3, $s \in irrational$. Thus $z \in irrational$, because of $z = sc$, with $c \in \mathbb{Z}^+$. So $z \notin \mathbb{Z}^+$.

The following example will explain in more detail the explanation of Theorem 3.5.

Example 3.6. We have the equation $1234567890^{34} + 230^{34} = z^{34}$. Based on Theorem 3.5., we get $(1234567890,230) = 10$. We have

$$(123456789.10)^{34} + (23.10)^{34} = (z_1.10)^{34}$$
$$\Leftrightarrow 123456789^{34} + 23^{34} = z_1^{34}$$

Based on Theorem 3.2, it applies $123456789 < z_1 < \left(123456789 + \frac{23}{34}\right)$, so that $z_1 \notin \mathbb{Z}^+$. Based on Definition 2.3, $z_1 \in irrational$. Thus $z \in irrational$, because of $z = z_1.10$. It is clear that $z \notin \mathbb{Z}^+$.

Example 3.7. We have the equation $243^3 + 18^3 = z^3$. Based on Theorem 3.5., we get $(243,18) = 9$. We have

$$(27.9)^3 + (2.9)^3 = (z_1.9)^3$$
$$\Leftrightarrow 27^3 + 2^3 = z_1^3$$

Based on Theorem 3.2, it applies $27 < z_1 < \left(27 + \frac{2}{3}\right)$, so that $z_1 \notin \mathbb{Z}^+$. Based on Definition 2.3, $z_1 \in irrational$. Thus $z \in irrational$, because of $z = z_1.9$. It is clear that $z \notin \mathbb{Z}^+$.

Next, we will consider the case of $q > n$ in equation $p^n + q^n = z^n$, with $p, q, n \in \mathbb{Z}^+$ and $q \leq p$.

Example 3.8. We have the equation $17^3 + 11^3 = z^3$. Based on Theorem 3.1, it applies

$$17 < z < \left(17 + \frac{11}{3}\right)$$
$$\Leftrightarrow 17 < z < 20\tfrac{2}{3}.$$

Note that there is a possibility $z = \{18,19,20\}$. For probability z like this, then we can try to substitute the value of z , and we will find z. But an experiment like this is ineffective.

Thus, in general, proof of condition $q > n$ has not been found for equation $p^n + q^n = z^n$.

4. Conclusion

In this study, we prove Fermat's Last Theorem for case $q \leq n$. In future research, we hope to prove The Fermat Last Theorem in the case of $q > n$.

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