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Co-Existing Point of Equilibrium in Discretization of Fractional-Order Prey and Predator Model

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A B S T R A C T

In this work, a discretization process of a fractional-order prey and predator model is discussed. The aim of this work is to describe the population phenomenon which contains prey and predator. In this research, the prey and predator model by Ghosh et al. (2017) is used. The model has an unique form because it contains prey refuge and additional food to predator. In order to give more details on prey and predator population, the model then modified into fractional order and then discretized. The discretization model has three points of equilibrium and one of them named co-existing point of equilibrium. The numerical simulation is used to perform the stability. The numerical simulation is controlled by using mathematical programming language. It resulted that the co-existing point of equilibrium tends to be stable or converge if a small value of s (time step) is selected. Otherwise, if a larger value of s is selected, then oscillatory is appeared which means the point of equilibrium become unstable or diverge.

Keywords: prey, predator, fractional-order, discretization, prey refuge, additional food.

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1. Introduction

In this 20-th century, the most popular study in world wide is how to manage or control a population which contain a lot of species (Prasad et al. 2013). A population with lot of variance species give the opportunity of predation to be appeared. Those species are separated into prey dan predator population.

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Firstly, around 19-th century, the predation of prey and predator phenomenon was researched by Lotka (Lotka, 1925) and Volterra (Volterra, 1931). The research has developed wider and bigger until now. Research in 2017, by Ghosh et al., presents prey and predator model with prey refuge and additional food for predator in the following model

$$\frac{dx}{dt} = x\left(1-\frac{x}{\gamma}\right) - \frac{(1-c')xy}{1+\alpha\xi+x}, \tag{1}$$

$$\frac{dy}{dt} = \frac{\beta[(1-c')x+\xi]y}{1+\alpha\xi+x} - \delta y. \tag{2}$$

Both parameter x and y represent the mass or population of prey and predator respectively.

In model (1) and (2), parameter c'denote a prey refuge where c' and other parameters are in interval $(0,1)$. A prey refuge is needed to protect the possibilty of prey extinction because of continuosly predation. The presence of prey refuge able to strengthen the stability of prey population (Wang, dkk. 2017). Some of refuge in real life such as a kangaroo rats avoid moonlight because bright light attract predator. Therefore, kangaroo rats being more quite in night to avoid their existence from predator (Jana et al., 2017).

The prey refuge has a negative effect on predator. Predator has to find another food to be consumed to meet their lifes needed. This another food is called the additional food for predator. The presence of this additional food affect positively in stabilitize predator population (Li, dkk. 2016). In model (1) and (2), additional food denotes in $\alpha\xi$ with α is the quality and ξ is quantity of additional food. Predation rate is symbolize with β and δ.

Model (1) and (2) produces three points of equilibrium and one of them is in form $x^* = \frac{\delta+\alpha\delta\xi-\beta\xi}{\beta(1-c')-\delta}$ and $y^* = \left(1-\frac{x^*}{\gamma}\right)\left(\frac{1+\alpha\xi+x^*}{1-c'}\right)$.

Prey and predator model is running in differential equation system. El Raheem et al. (2014) investigates latest research is mostly in fractional-order. Latest research use fractional-order because its ability to explain the phenomenon in the past and current situation more detail and accurate. A memory is using in fractional-order (Elsadany, 2015).

Fractional-order has several definitions and one of them, the famous one, is Caputo fractional derrivative (Petras, 2011) in form

$${}_a^C D_t^n\, u(t) = \frac{1}{\Gamma(m-n)} \int_a^t \frac{u^m(\tau)}{(t-\tau)^{n-m+1}}\, d\tau. \tag{3}$$

Nature phenomenon is more realistic when it is in fractional-order. El Shahed (2016) explained that fractional-order is not only explore its dynamical behavior, but also provide information on the impact of its dynamic behavior in the present and in the future, in more detail and realistic. Some rare project of scientist in work of fractional-order is develop it into discrete form (Miller and Ross, 1988).

In this work, a discrete fractional-order of prey and predator model (1) and (2) is discussed. The goal is to explain the population according to numerical simulations.

2. The Discretization Process and Point of Equilibrium

2.1. The discretization process.

Model (1) is modified into the following.

$$D_t^n \mathrm{x} = x(t)\left(1-\frac{x(t)}{\gamma}\right) - \frac{(1-c')x(t)y(t)}{1+\alpha\xi+x(t)}, \tag{4}$$

$$D_t^n \mathrm{y} = \frac{\beta[(1-c')x(t)+\xi]y(t)}{1+\alpha\xi+x(t)} - \delta y(t). \tag{5}$$

By substitute $\frac{dx}{dt}$ and $\frac{dy}{dt}$ form into $D_t^n\mathrm{x}$ and $D_t^n\mathrm{y}$ form, the discretization process is begun. Based on Elsadany and Matouk (2015) did on their paper, model (4) and (5) nextly can be modified into

$$\mathrm{x}_{m+1} = x_m + \frac{s^n}{n\Gamma(n)}\left(x_m\left(1-\frac{x_m}{\gamma}\right)\right) - \frac{s^n}{n\Gamma(n)}\left(\frac{(1-c')x_m y_m}{1+\alpha\xi+x_m}\right), \tag{4}$$

$$y_{m+1} = \mathrm{y_m} - \frac{s^n}{n\Gamma(n)}(\delta y_m) + \frac{s^n}{n\Gamma(n)}\left(\frac{\beta[(1-c')x_m+\xi]y_m}{1+\alpha\xi+x_m}\right). \tag{5}$$

2.2. The point of equilibrium

If the growth rate of population is zero, the point of equilibrium is then obtained. The moment the point of equilibrium is obtained, then the numerical simulation can be perfomed.

The point of equilibrium of model (4) and (5) is x^* and y^* if it satisfies

$$x^* = x^* + \frac{s^n}{n\Gamma(n)}\left(x^*\left(1-\frac{x^*}{\gamma}\right)\right) - \frac{s^n}{n\Gamma(n)}\left(\frac{(1-c')x^*y^*}{1+\alpha\xi+x^*}\right), \qquad (6)$$

$$y^* = y^* - \frac{s^n}{n\Gamma(n)}(\delta y^*) + \frac{s^n}{n\Gamma(n)}\left(\frac{\beta[(1-c')x^*+\xi]y^*}{1+\alpha\xi+x^*}\right). \qquad (7)$$

Model (6) and (7) has the points of equilibrium and one of them is called the co-existing point of equilibrium, that is $x^* = \frac{\delta+\alpha\delta\xi-\beta\xi}{\beta(1-c')-\delta}$ and $y^* = \left(1-\frac{x^*}{\gamma}\right)\left(\frac{1+\alpha\xi-x^*}{1-c'}\right)$

3. Numerical Simulation

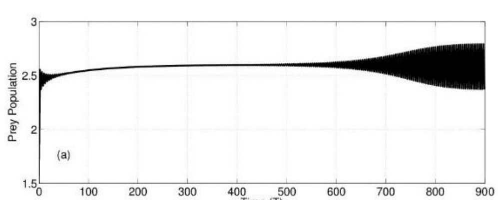
The numerical simulation is perfomed using parameters in Table 1.

Table 1 – Parameters

Parameters	Value
α	0,6
γ	2,6
c'	0,6
ξ	0,2
δ	0,08
β	0,21
n	0,9

Based on Table 1, the numerical simulation is ploted. In this case, a few different value of s is chosen. Parameter s indicates a time step.

Figure 1 shows that oscillates appears when x is around 2,6 on prey population. In this case, the prey population is tend to be unstable. Osscilate means the value does not converge to one point. For all figures, prey and predator mass are in thousands and time (t) is in days.



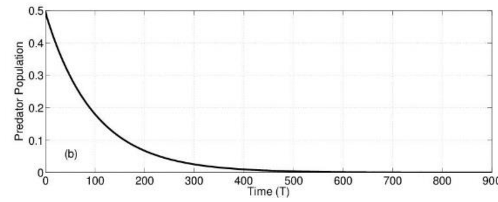


Figure. 1 – Numerical simulation using $s = 2,1$.

Next, a different time step (s) is chosen. By replacing $s = 2,1$ into $s = 4$, Figure 2 is ploted.

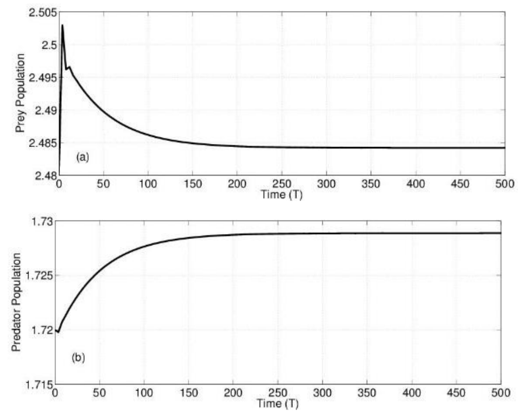


Figure. 2 – Numerical simulation using $s = 4$.

In Figure 2, value of $s = 4$ is chosen. Figure 4 shows that each population of prey and predator tends to asymptotically stable and converge to co-existing point of equilibrium that is 2,48 and 1,72 for both prey and predator respectively. Figure 3 gives more accurate and detail component. Figure 3 using $s =$ 6,8.

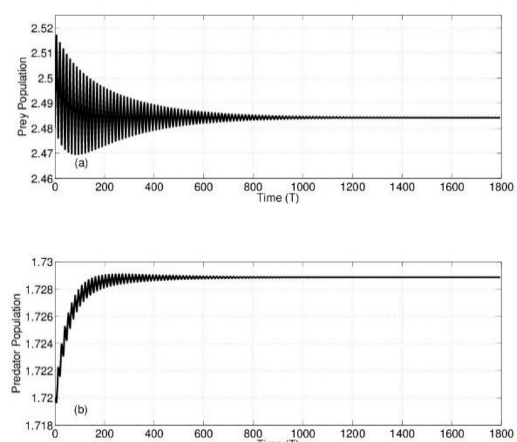


Figure. 3 – Numerical simulation using $s = 6, 8$.

The last, if a larger value of s is chosen, Figure 4 shows that both prey and predator population is damage or blowing up. The value does not converge to one point. Otherwise, the value is diverge.

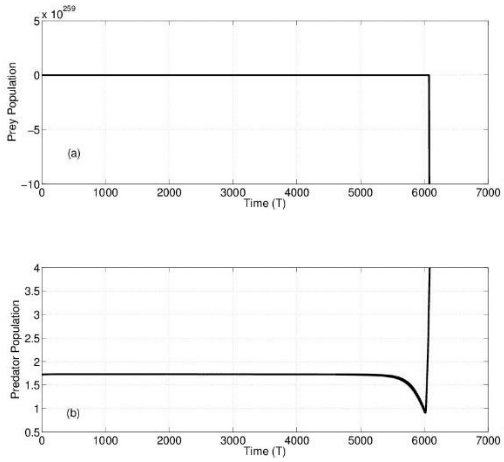


Figure. 4 – Numerical simulation using $s = 6, 91$.

4. Conclusion

In this work, a discrete fractional-order of prey and predator model (1) and (2) is discussed. The model has three points of equilibrium and one of them is co-existing point of equilibrium. The numerical simulation showed that the population on both prey and predator depends on the value of time step (s) that chosen. The larger the value of s is chosen, the quicker the population to be blowing up. If smaller value of s is selected, the point of equilibirum tends to be stable or convergent. In the future, the other point of equilibrium will be explored.

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