



## Numerical Solution of the Korteweg-De Vries Equation Using Finite Difference Method

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### ABSTRACT

The Korteweg-de Vries (KdV) equation is a nonlinear partial differential equation that has a key role in wave physics and many other disciplines. In this article, we develop numerical solutions of the KdV equation using the finite difference method with the Crank-Nicolson scheme. We explain the basic theory behind the KdV equation and the finite difference method, and outline the implementation of the Crank-Nicolson scheme in this context. We also give an overview of the space and time discretization and initial conditions used in the simulation. The results of these simulations are presented through graphical visualizations, which allow us to understand how the KdV solution evolves over time. Through analysis of the results, we explore the behavior of the solutions and perform comparisons with exact solutions in certain cases. Our conclusion summarizes our findings and discusses the advantages and limitations of the method used. We also provide suggestions for future research in this area.

**Keywords:** Equation KdV, Soliton, Finite difference method, Crank-nicolson scheme

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### 1. Introduction

Differential Partial Equations are widely used in mathematical physics problem. The solution can be founded analytically, but the most of partial differential equations are difficult to solve. Then, it needs a numerical method to find the solutions. One of example is Nonlinear Schrödinger Equations by using Petviashvili method (Nuzla, 2022). Another example of differential partial equations is the Korteweg-de Vries (KdV). KdV equation is a nonlinear partial differential equation used as a mathematical model for waves on the surface of shallow water. It was first introduced by Korteweg and de Vries in 1895 as a mathematical model for water waves in a flat and wide channel (Suryanto, 2017). The KdV equation is a nonlinear equation for which numerical solutions can be found both analytically and numerically. It is a nonlinear wave equation, and its solutions are not always obtainable exactly (Butcher, 2008). The KdV equation is also known as a prototype of a model that can be solved exactly, meaning it allows for precise and accurate determination of its solutions (Causon & Mingham, 2010).

The Korteweg-de Vries equation has three terms: the nonlinear term, the dissipative term, and the term involving the time derivative ( $t$ ). In general, the Korteweg-de Vries equation can be written as follows.

$$u_t + uu_x + su_{xxx} = 0 \quad (1.1)$$

where  $uu_x$  is the nonlinear term, and  $su_{xxx}$  is the dissipative term. Equation (1.1) has two independent variables,  $x$  and  $t$ , as well as the dependent variable  $u$ , which represents the wave amplitude.

The solutions to the Korteweg-De Vries equation can be obtained through various methods, both analytically and numerically. Numerical solutions of the KdV equation can be found using methods such as spectral methods, finite difference methods, differential transformation methods, and others (Djojodihardjo, 2000). Meanwhile, for analytical solutions of the Korteweg-De Vries equation, the solutions are obtained as follows.

$$u(x, t) = \frac{2sk^2}{\alpha} \operatorname{sech}^2(k(x - 42sk^2t)). \quad (1.2)$$

The above analytical solution describes a soliton propagating along  $t$ . Based on the analytical solution, it is known that the dispersive coefficient affects the height and speed of soliton propagation, with larger values of  $s$  resulting in higher and faster propagation (Daxois & Michel, 2010). The finite difference method is one of numerical method for solving ordinary and partial differential equations. This method can describe the stable solution and convergence to

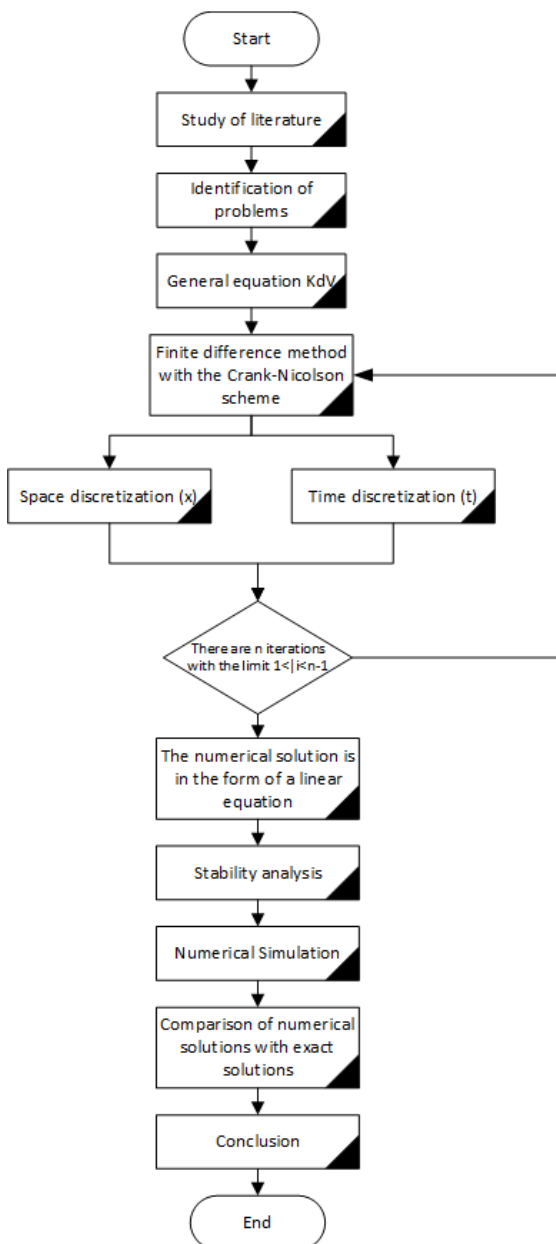
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the analytical solution. The finite difference method is based on Taylor series expansion, which leads to backward, forward, and central difference equations (Hasan, et al, 2016).

This research will discuss the numerical solution of the Kortweg-De Vries equation using the finite difference method. The numerical solution of the Kortweg-De Vries equation will be aided by analytical tools using *Wolfram Mathematica 13 for Windows* and *MATLAB software*.

## 2. Methodology

In this section, the method used to solve the problem of the numerical solution of the Kortweg-De Vries equation using the finite difference method will be explained. This research is based on a previous article about the numerical solution of the Kortweg-De Vries equation using the spectral method (Yin, et al, 2023). In this study, *Wolfram Mathematica 13 for Windows software* is used.



**Figure 3.1** Flowchart of how to find the numerical solution of KDV Equation

### 2.1 Finite Difference Method

The selection of the method to solve the problem of the numerical solution of the Kortweg-De Vries equation is the finite difference method, and it will be assisted by the use of mathematical analysis software, *Wolfram Mathematica 13 for Windows* and *POM QM*. The finite difference method is a commonly used numerical method for solving technical issues and mathematical problems related to physical phenomena (Munir, 2010). Its principle involves replacing the derivatives in a differential equation with finite difference discretization based on Taylor series. Physically, the Taylor series can be interpreted as a quantity observed in a certain space and time (observed space and time) that can be calculated from the same quantity at specific space and time points, with small differences in observed space and time (Rumlawang, 2013).

If  $f$  and all its derivatives  $f', f'', f''', \dots$ , exist in the interval  $[a, b]$ , then the value of  $f(x)$  for  $x$  around  $x_0$  and  $x \in [a, b]$  can be expanded into a Taylor series (Suryanto, 2017):

$$f(x) = f(x_0) + \frac{(x-x_0)}{1!}f'(x_0) + \frac{(x-x_0)^2}{2!}f''(x_0) + \dots + \frac{(x-x_0)^m}{m!}f^m(x_0) + \dots \quad (2.1)$$

If  $x - x_0 = h$ , then  $f(x)$  is:

$$f(x) = f(x_0) + \frac{h}{1!}f'(x_0) + \frac{h^2}{2!}f''(x_0) + \dots + \frac{(h)^m}{m!}f^m(x_0) + \dots \quad (2.2)$$

Several finite difference schemes will be explained as follows:

#### 1. Forward Scheme

By using the first three terms on the right-hand side of the Taylor series in equation (2.1), we obtain:

$$f^{(1)}(x_i) = \frac{f(x_i + \Delta x) - f(x_i)}{\Delta x} - \frac{\Delta x}{2!}f^{(2)}(x_i)$$

$$\left. \frac{df}{dx} \right|_{x=x_i} = \frac{f(x_i + \Delta x) - f(x_i)}{\Delta x} - \frac{\Delta x}{2!}f^{(2)}(x_i) \quad (2.3)$$

$$\left. \frac{df}{dx} \right|_{x=x_i} = \frac{f(x_i + \Delta x) - f(x_i)}{\Delta x}$$

From equation (2.3), the forward scheme is said to have a first-order error or  $\delta x(\Delta x)$ .

As indicated by the forward scheme using a finite grid, the forward scheme (Lestari, 2013):

#### a. Spatial Finite Difference:

$$\left. \frac{df}{dx} \right|_i = \frac{f_{i+1}^n - f_i^n}{\Delta x} \quad \text{or} \quad \frac{\partial f}{\partial x} = \frac{f_{i+1}^{n+1} - f_i^{n+1}}{\Delta x}$$

with  $\Delta x = x_{i+1} - x_i$ .

- b. Finite Difference in Time ( Finite Difference Depends on Time) can be implemented using one of the following discretization methods:

$$\frac{df}{dt}\Big|_i = \frac{f_i^{n+1} - f_i^n}{\Delta t}$$

or

$$\frac{\partial f}{\partial t}\Big|_{i+1} = \frac{f_{i+1}^{n+1} - f_{i+1}^n}{\Delta t}$$

## 2. Backward Scheme

By using the first three terms from the right-hand side of the Taylor series, we obtain Equation (2.4):

$$f(x_i + \Delta x) = f(x_i) - \frac{(\Delta x)^2}{1!} f^{(1)}(x_i) + \frac{(\Delta x)^2}{2!} f^{(2)}(x_i)$$

$$f^{(1)}(x_i) = \frac{f(x_i) - f(x_i - \Delta x)}{2!} - \frac{\Delta x}{2!} f^{(2)}(x_i)$$

$$\frac{df}{dx}\Big|_{x=x_i} = \frac{f(x_i) - f(x_i - \Delta x)}{2!} - \frac{\Delta x}{2!} f^{(2)}(x_i) \quad (2.4)$$

$$\frac{df}{dx}\Big|_{x=x_i} = \frac{f(x_i) - f(x_i + \Delta x)}{\Delta x}$$

In the forward scheme, information at the computation point  $i$  is connected to the information at the point  $i - 1$  located in front of it, as explained in the backward scheme:

As shown in the scheme using the grid of finite differences, the backward scheme can be seen at  $i$ :

- a. Finite differences in space:

$$\frac{df}{dx}\Big|_i = \frac{f_i^n - f_{i-1}^n}{\Delta x} \text{ or } \frac{\partial f}{\partial x}\Big|_i = \frac{f_i^{n+1} - f_{i-1}^{n+1}}{\Delta x}$$

- b. While finite differences in time:

$$\frac{df}{dt}\Big|_{i-1} = \frac{f_{i-1}^{n+1} - f_{i-1}^n}{\Delta t} \text{ or } \frac{\partial f}{\partial t}\Big|_i = \frac{f_i^{n+1} - f_i^n}{\Delta t}$$

## 3. Central Scheme

If the Taylor series from equation (2.3) is subtracted from the Taylor series from equation (2.4), the central scheme will be obtained.

As shown by the central scheme using a finite difference grid, the central scheme:

- a. Finite Differences in Space:

$$\frac{df}{dx}\Big|_i = \frac{f_{i+1}^n - f_{i-1}^n}{2\Delta x} \text{ or } \frac{\partial f}{\partial x}\Big|_i = \frac{f_{i+1}^{n+1} - f_{i-1}^{n+1}}{2\Delta x}$$

$$\frac{\partial^2 f}{\partial x^2}\Big|_i = \frac{\partial}{\partial x} \left( \frac{\partial f}{\partial x} \right) = \frac{\left( \frac{\partial f}{\partial x} \right)_{forward} - \left( \frac{\partial f}{\partial x} \right)_{backward}}{\Delta x}$$

$$= \frac{\frac{f_{i+1} - f_i}{\Delta x} + \frac{f_{i+1} - f_i}{\Delta x}}{(\Delta x)^2} = \frac{f_{i+1} - 2f_i + f_{i-1}}{(\Delta x)^2}$$

for  $t^{n+1}$  will be

$$\frac{\partial^2 f}{\partial x^2}\Big|_i = \frac{f_{i+1}^{n+1} - 2f_i^{n+1} + f_{i-1}^{n+1}}{(\Delta x)^2}$$

- b. Finite differences in time:

$$\frac{df}{dt}\Big|_{i-1} = \frac{f_{i-1}^{n+1} - f_{i-1}^n}{\Delta x}, \frac{\partial f}{\partial t}\Big|_i = \frac{f_i^{n+1} - f_i^n}{\Delta x}$$

$$= \frac{\partial f}{\partial t}\Big|_{i+1} = \frac{f_{i+1}^{n+1} - f_{i+1}^n}{\Delta x}$$

with  $\Delta x = t^{n+1} - t^n$ .

## 4. Leap-Frog Scheme (Leap-Frog)

The leap-frog method is one of the finite difference methods that employs the central finite difference approach (Strauss, 2007).

As shown in the scheme above, using a finite difference grid, the leap-frog scheme can be written as follows:

- a. Finite differences in space:

$$\frac{df}{dx}\Big|_i = \frac{f_{i+1}^n - f_{i-1}^n}{2\Delta x}$$

- b. Finite differences in time:

$$\frac{df}{dx}\Big|_i = \frac{f_i^{n+1} - f_i^{n-1}}{2\Delta x}$$

## 5. Dufort-Frankel Scheme

Using a finite difference grid, the Dufort-Frankel scheme can be expressed as follows:

- a. Finite differences in space:

$$\frac{\partial^2 f}{\partial x^2}\Big|_i = \frac{f_{i+1}^n - f_i^{n+1} + f_{i-1}^{n-1}}{(\Delta x)^2}$$

- b. Finite differences in time:

$$\frac{df}{dt}\Big|_i = \frac{f_i^{n+1} - f_i^{n-1}}{2\Delta t}$$

## 6. Crank-Nicolson Scheme

This scheme uses weighting techniques for discretizing the current time ( $t^n$ ) and the future time ( $t^{n+1}$ ) in a more flexible way, employing time weighting factors.

- a. Finite differences in space:

$$\frac{\partial^2 f}{\partial x^2}\Big|_i = \hat{\epsilon} \left( \frac{f_{i+1}^{n+1} - 2f_i^{n+1} + f_{i-1}^{n+1}}{(\Delta x)^2} \right) + (1 - \hat{\epsilon}) \left( \frac{f_{i+1}^n - 2f_i^n + f_{i-1}^n}{(\Delta x)^2} \right)$$

with  $0 \leq \hat{\epsilon} \leq 1$  is the time weighting factor.

- b. Finite differences in time:

$$\frac{\partial f}{\partial t}\Big|_i = \frac{f_i^{n+1} - f_i^n}{\Delta t}$$

## 2.2 Data Processing and Analysis

Data processing and analysis are carried out using the finite difference method for the numerical solution of the Kortweg-De Vries equation with the help of analytical tools

using Wolfram Mathematica 13 for Windows and POM QM software.

### 3. Result and Discussion

In this section, we will discuss the use of finite difference methods in determining the numerical solution of the KdV equation. The numerical solutions obtained are expected to provide references for developing numerical codes for the KdV equation to achieve effective and efficient solutions. The KdV equation model used in this discussion is the general KdV equation that has been used in previous research. The equation used is as follows (Zhang, et al, 2014):

$$u_t + uu_x + u_{xxx} = 0 \quad (3.1)$$

here,  $u$  is a function representing waves or solitons,  $t$  is time, and  $x$  is position in space. The KdV equation is used in various contexts, including the study of soliton waves in water or other media.

In this section, we will discuss determining the numerical solution of the KdV equation using the finite difference method with the Crank-Nicolson scheme. The finite difference method is a commonly used technique for numerically solving partial differential equations such as the KdV equation.

#### 3.1 Discretization of Space and Time

In this part, the  $x$ -domain is divided into a number of grid points (discrete points) with a distance between grid points  $\Delta x$ . The time domain  $t$  is also divided into discrete intervals with a distance  $\Delta t$ . In this section, the solution of the KdV equation using the Crank-Nicolson scheme will be explained. Discretization of time  $t$  is done first to simplify the subsequent steps. The results of the discretization are as follows:

$$u_t \approx \frac{u_i^{s+1} - u_i^s}{\Delta t} \quad (3.2)$$

$$u \approx \frac{u_i^s + u_i^{s+1}}{2} \quad (3.3)$$

$$u_x \approx \frac{u_{i+1}^{s+1} - u_{i-1}^{s+1} + u_{i+1}^s - u_{i-1}^s}{4\Delta x^2} \quad (3.4)$$

$$\frac{u_{i+2}^{s+1} - 2u_{i+1}^{s+1} + 2u_{i-1}^{s+1} - u_{i-2}^{s+1} + u_{i+2}^s - 2u_{i+1}^s + 2u_{i-1}^s - u_{i-2}^s}{4\Delta x^3} \quad (3.5)$$

Next, equations (3.2), (3.3), (3.4), (3.5) are substituted into the KdV equation (3.1). Based on the Crank-Nicolson scheme, the values of  $u_t$ ,  $u$ ,  $u_x$ , and  $u_{xxx}$  are combined on one side of the equation to simplify the calculation, resulting in:

$$vu_{i+2}^{s+1} - 2vu_{i+1}^{s+1} + \left(1 + \frac{1}{2}v\Delta x^2(u_{i+1}^{s+1} - u_{i-1}^{s+1} + u_{i+1}^s - u_{i-1}^s)\right)u_i^{s+1} + 2vu_{i-1}^{s+1} - vu_{i-2}^{s+1} = -vu_{i+2}^s + 2vu_{i+1}^s +$$

$$\left(1 - \frac{1}{2}v\Delta x^2(u_{i+1}^{s+1} - u_{i-1}^{s+1} + u_{i+1}^s - u_{i-1}^s)\right)u_i^s - 2vu_{i-1}^s + vu_{i-2}^s \quad (3.6)$$

with the value  $v$  being  $\frac{\Delta t}{\Delta x^2}$ , based on the obtained calculations, the solution to the KdV equation can be determined through several iterations. The number of iterations used is  $n$  iterations with the condition  $1 \leq i \leq n-1$ . By applying these conditions, the matrix obtained is as follows:

$$\begin{bmatrix} c_1 & d_1 & e_1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ b_2 & c_2 & d_2 & e_2 & 0 & \cdots & \cdots & \cdots & 0 \\ a_3 & b_3 & c_3 & d_3 & e_3 & 0 & \cdots & \cdots & 0 \\ 0 & a_4 & b_4 & c_4 & d_4 & e_4 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & & & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & a_{n-4} & b_{n-4} & c_{n-4} & d_{n-4} & e_{n-4} & 0 \\ 0 & \cdots & \cdots & 0 & a_{n-3} & b_{n-3} & c_{n-3} & d_{n-3} & e_{n-3} \\ 0 & \cdots & \cdots & \cdots & 0 & a_{n-2} & b_{n-2} & c_{n-2} & d_{n-2} \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 & a_{n-1} & b_{n-1} & c_{n-1} \end{bmatrix} \begin{bmatrix} U_1^{s+1} \\ U_2^{s+1} \\ U_3^{s+1} \\ U_4^{s+1} \\ \vdots \\ U_{n-4}^{s+1} \\ U_{n-3}^{s+1} \\ U_{n-2}^{s+1} \\ U_{n-1}^{s+1} \end{bmatrix}$$

$$\begin{bmatrix} \tilde{c}_1 & \tilde{d}_1 & \tilde{e}_1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ \tilde{b}_2 & \tilde{c}_2 & \tilde{d}_2 & \tilde{e}_2 & 0 & \cdots & \cdots & \cdots & 0 \\ \tilde{a}_3 & \tilde{b}_3 & \tilde{c}_3 & \tilde{d}_3 & \tilde{e}_3 & 0 & \cdots & \cdots & 0 \\ 0 & \tilde{a}_4 & \tilde{b}_4 & \tilde{c}_4 & \tilde{d}_4 & \tilde{e}_4 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & & & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & \tilde{a}_{n-4} & \tilde{b}_{n-4} & \tilde{c}_{n-4} & \tilde{d}_{n-4} & \tilde{e}_{n-4} & 0 \\ 0 & \cdots & \cdots & 0 & \tilde{a}_{n-3} & \tilde{b}_{n-3} & \tilde{c}_{n-3} & \tilde{d}_{n-3} & \tilde{e}_{n-3} \\ 0 & \cdots & \cdots & \cdots & 0 & \tilde{a}_{n-2} & \tilde{b}_{n-2} & \tilde{c}_{n-2} & \tilde{d}_{n-2} \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 & \tilde{a}_{n-1} & \tilde{b}_{n-1} & \tilde{c}_{n-1} \end{bmatrix} \begin{bmatrix} U_1^s \\ U_2^s \\ U_3^s \\ U_4^{s+1} \\ \vdots \\ U_{n-4}^s \\ U_{n-3}^s \\ U_{n-2}^s \\ U_{n-1}^s \end{bmatrix}$$

With:

$$a_j = -v, j = 3, 4, \dots, n-1$$

$$b_j = 2v, j = 2, 3, \dots, n-1$$

$$c_i = 1 + \frac{1}{2}v\Delta x^2(U_2^{s+1} + U_2^s) - v$$

$$c_j = 1 + \frac{1}{2}v\Delta x^2(U_{j+1}^{s+1} - U_{j-1}^{s+1} + U_{j+1}^s - U_{j-1}^s), j = 2, 3, \dots, n-2$$

$$c_{n-1} = 1 + \frac{1}{2}v\Delta x^2(-U_{n-2}^{s+1} + U_{n-2}^s) + v.$$

$$d_j = -2v, j = 1, 2, \dots, n-2$$

$$e_j = v, j = 1, 2, \dots, n-3$$

$$\tilde{a}_j = v, j = 3, 4, \dots, n-1$$

$$\tilde{b}_j = -v, j = 2, 3, \dots, n-1$$

$$\tilde{c}_1 = 1 + \frac{1}{2}v\Delta x^2(U_2^{s+1} + U_2^s) - v$$

$$\tilde{c}_j = 1 + \frac{1}{2}v\Delta x^2(U_{j+1}^{s+1} - U_{j-1}^{s+1} + U_{j+1}^s - U_{j-1}^s), j = 2, 3, \dots, n-2$$

$$\tilde{c}_{n-1} = 1 + \frac{1}{2}v\Delta x^2(-U_{n-2}^{s+1} + U_{n-2}^s) + v.$$

$$\tilde{d}_j = 2v, j = 1, 2, \dots, n-2$$

$$\tilde{e}_j = -v, j = 1, 2, \dots, n-3$$

Based on the matrix description above, a system of equations can be formulated as follows:

$$A\bar{U}_i^{s+1} = B\bar{U}_i^{s+1} \quad (3.7)$$

where  $A$  is coefficient matrix  $n - 1 \times n - 1$  and  $B$  is coefficient matrix  $n - 1 \times n - 1$ ,  $\bar{U}_i^s$  is constant vector  $n - 1 \times 1$  and  $\bar{U}_i^{s+1}$  is vector solution.

### 3.2 Stability Analysis

The stability analysis is important to determine whether the method used in this research is stable or not. In this analysis, we use the von Neumann stability analysis by substituting  $u_j^n = \rho^n e^{iaj}$ , with  $i = \sqrt{-1}$  into the discretized equation. However, equation (3.1) must be linearized to obtain:

$$u_t + \bar{u}u_x + u_{xxx} = 0 \quad (3.8)$$

Furthermore, equation (3.11) is discretized with respect to  $t$  and  $x$ , resulting in:

$$\begin{aligned} & \frac{U_j^{n+1} - U_j^n}{\Delta t} + \frac{1}{2} \left( \frac{U_{j+1}^{n+1} - U_{j-1}^{n+1}}{2\Delta x} \right) + \frac{1}{2} \left( \frac{U_{j+1}^n - U_{j-1}^n}{2\Delta x} \right) \\ & + \frac{1}{2} \left[ U \left( \frac{U_{j+1}^{n+1} - U_{j-1}^{n+1}}{2\Delta x} \right) \right] + \frac{1}{2} \left[ U \left( \frac{U_{j+1}^n - U_{j-1}^n}{2\Delta x} \right) \right] \\ & + \frac{1}{2} \left( \frac{U_{j+2}^{n+1} - 2U_{j+1}^{n+1} + 2U_{j-1}^{n+1} - 2U_{j-2}^{n+1}}{2\Delta x^3} \right) \\ & + \frac{1}{2} \left( \frac{U_{j+2}^n - 2U_{j+1}^n + 2U_{j-1}^n - 2U_{j-2}^n}{2\Delta x^3} \right) = 0 \end{aligned}$$

Next, the result is simplified to:

$$\begin{aligned} & \frac{U_j^{n+1}}{\Delta t} + \frac{1}{2} \left( \frac{U_{j+1}^{n+1} - U_{j-1}^{n+1}}{2\Delta x} \right) + \frac{1}{2} \left[ \bar{U} \left( \frac{U_{j+1}^{n+1} - U_{j-1}^{n+1}}{2\Delta x} \right) \right] \\ & + \frac{1}{2} \left( \frac{U_{j+2}^{n+1} - 2U_{j+1}^{n+1} + 2U_{j-1}^{n+1} - 2U_{j-2}^{n+1}}{2\Delta x^3} \right) \\ & = \frac{U_j^n}{\Delta t} + \frac{1}{2} \left( \frac{U_{j+1}^n - U_{j-1}^n}{2\Delta x} \right) + \frac{1}{2} \left[ \bar{U} \left( \frac{U_{j+1}^n - U_{j-1}^n}{2\Delta x} \right) \right] \\ & + \frac{1}{2} \left( \frac{U_{j+2}^n - 2U_{j+1}^n + 2U_{j-1}^n - 2U_{j-2}^n}{2\Delta x^3} \right) \end{aligned}$$

Then, substituting the value of  $U_j^n = \rho^n e^{iaj}$ ,  $i = \sqrt{-1}$  into the equation, we get:

$$\begin{aligned} & \frac{\rho^{n+1} e^{iaj}}{\Delta t} + \frac{1}{2} \left( \frac{\rho^{n+1} e^{ia(j+1)} - \rho^{n+1} e^{ia(j-1)}}{2\Delta x} \right) \\ & + \frac{1}{2} \left[ \bar{U} \left( \frac{\rho^{n+1} e^{ia(j+1)} - \rho^{n+1} e^{ia(j-1)}}{2\Delta x} \right) \right] \\ & + \frac{1}{2} \left( \frac{\rho^{n+1} e^{ia(j+2)} - \rho^{n+1} e^{ia(j+1)} + \rho^{n+1} e^{ia(j-1)} - \rho^{n+1} e^{ia(j-2)}}{2\Delta x^3} \right) \\ & = \frac{\rho^n e^{iaj}}{\Delta t} + \frac{1}{2} \left( \frac{\rho^{n+1} e^{ia(j+1)} - \rho^{n+1} e^{ia(j-1)}}{2\Delta x} \right) \\ & + \frac{1}{2} \left[ \bar{U} \left( \frac{\rho^{n+1} e^{ia(j+1)} - \rho^{n+1} e^{ia(j-1)}}{2\Delta x} \right) \right] \\ & + \frac{1}{2} \left( \frac{\rho^{n+1} e^{ia(j+2)} - \rho^{n+1} e^{ia(j+1)} + \rho^{n+1} e^{ia(j-1)} - \rho^{n+1} e^{ia(j-2)}}{2\Delta x^3} \right) \end{aligned}$$

Next, by dividing  $\rho^n e^{iaj}$  in order to get a simpler equation, we obtain:

$$\begin{aligned} & \frac{\rho}{\Delta t} + \frac{1}{2} \left( \frac{\rho e^{ia} - \rho e^{-ia}}{2\Delta x} \right) - \frac{1}{2} \left[ \bar{U} \left( \frac{\rho e^{ia} - \rho e^{-ia}}{2\Delta x} \right) \right] \\ & - \frac{1}{2} \left( \frac{\rho e^{2ia} - \rho e^{ia} + \rho e^{-ia} - \rho e^{-2ia}}{2\Delta x^3} \right) \\ & = \frac{1}{\Delta t} - \frac{1}{2} \left( \frac{e^{ia} - e^{-ia}}{2\Delta x} \right) \\ & + \frac{1}{2} \left[ \bar{U} \left( \frac{e^{ia} - e^{-ia}}{2\Delta x} \right) \right] \\ & + \frac{1}{2} \left( \frac{e^{2ia} - e^{ia} + e^{-ia} - e^{-2ia}}{2\Delta x^3} \right) \end{aligned}$$

From  $e^{\pm ia} = \cos(a) \pm i \sin(a)$ , we have:

$$\begin{aligned} & \frac{\rho}{\Delta t} + \frac{1}{2} \left( \frac{\rho(\cos(a) + i \sin(a)) - \rho(\cos(a) - i \sin(a))}{2\Delta x} \right) \\ & - \frac{1}{2} \left[ \bar{U} \left( \frac{\rho(\cos(a) + i \sin(a)) - \rho(\cos(a) - i \sin(a))}{2\Delta x} \right) \right] \frac{1}{\Delta t} \\ & + \frac{1}{2} \left( \frac{(\cos(a) + i \sin(a)) - (\cos(a) - i \sin(a))}{2\Delta x} \right) \end{aligned}$$

With algebraic calculation, we obtain:

$$\begin{aligned} & \left[ 1 + i\Delta t \left[ \left( \frac{\sin(a) - \bar{U} \sin(a)}{2\Delta x} \right) - \left( \frac{\sin(2a) - 2 \sin(a)}{2\Delta x} \right) \right] \right] \rho \\ & - \left[ 1 \right. \\ & \left. - i\Delta t \left[ \left( \frac{\sin(a) - \bar{U} \sin(a)}{2\Delta x} \right) \right. \right. \\ & \left. \left. - \left( \frac{\sin(2a) - 2 \sin(a)}{2\Delta x} \right) \right] \right] = 0 \end{aligned}$$

So, the value of  $\rho$  can be determined as:

$$\rho = \frac{1 + \left\{ -i\Delta t \left[ \left( \frac{\sin(a) - \bar{U} \sin(a)}{2\Delta x} \right) - \left( \frac{\sin(2a) - 2 \sin(a)}{2\Delta x} \right) \right] \right\}}{1 + i\Delta t \left[ \left( \frac{\sin(a) - \bar{U} \sin(a)}{2\Delta x} \right) - \left( \frac{\sin(2a) - 2 \sin(a)}{2\Delta x} \right) \right]}$$

Next, calculate  $|\rho|$  as follows:

$$\begin{aligned} & |\rho| \\ & = \left| \frac{1 + \left\{ -i\Delta t \left[ \left( \frac{\sin(a) - \bar{U} \sin(a)}{2\Delta x} \right) - \left( \frac{\sin(2a) - 2 \sin(a)}{2\Delta x} \right) \right] \right\}}{1 + i\Delta t \left[ \left( \frac{\sin(a) - \bar{U} \sin(a)}{2\Delta x} \right) - \left( \frac{\sin(2a) - 2 \sin(a)}{2\Delta x} \right) \right]} \right| \end{aligned}$$

We have  $\left| \frac{z_1}{z_2} \right| = \left| \frac{z_1}{z_2} \right|$   $z_1 = x + iy$  and  $|z| = \sqrt{x^2 + y^2}$ , so:

$$\begin{aligned} & |\rho| \\ & = \frac{\left| 1 + \left\{ -i\Delta t \left[ \left( \frac{\sin(a) - \bar{U} \sin(a)}{2\Delta x} \right) - \left( \frac{\sin(2a) - 2 \sin(a)}{2\Delta x} \right) \right] \right\} \right|}{\left| 1 + \left\{ i\Delta t \left[ \left( \frac{\sin(a) - \bar{U} \sin(a)}{2\Delta x} \right) - \left( \frac{\sin(2a) - 2 \sin(a)}{2\Delta x} \right) \right] \right\} \right|} \\ & = \frac{\sqrt{1^2 + \left\{ -i\Delta t \left[ \left( \frac{\sin(a) - \bar{U} \sin(a)}{2\Delta x} \right) - \left( \frac{\sin(2a) - 2 \sin(a)}{2\Delta x} \right) \right] \right\}^2}}{\sqrt{1^2 + \left\{ i\Delta t \left[ \left( \frac{\sin(a) - \bar{U} \sin(a)}{2\Delta x} \right) - \left( \frac{\sin(2a) - 2 \sin(a)}{2\Delta x} \right) \right] \right\}^2}} \end{aligned}$$

$$|\rho| = \frac{\sqrt{1^2 + \left\{ i\Delta t \left[ \left( \frac{\sin(a) - \bar{U} \sin(a)}{2\Delta x} \right) - \left( \frac{\sin 2(a) - 2 \sin(a)}{2\Delta x} \right) \right] \right\}^2}}{\sqrt{1^2 + \left\{ i\Delta t \left[ \left( \frac{\sin(a) - \bar{U} \sin(a)}{2\Delta x} \right) - \left( \frac{\sin 2(a) - 2 \sin(a)}{2\Delta x} \right) \right] \right\}^2}} = 1$$

The final result indicates  $|\rho| = 1$ , so it can be concluded that the method used is unconditionally stable and allows freedom in choosing  $\Delta t$  and  $\Delta x$  for simulations.

### 3.3 Simulation and Interpretation of Results using Wolfram Mathematica Software

The discretization results obtained for the KdV equation in Equation (3.6) serve as the initial step in determining the numerical solution. These discretization results are then subjected to simulation to verify whether the applied method yields an appropriate solution. The calculations performed during this simulation require the assistance of computer software to facilitate the process. The software used is *Wolfram Mathematica 13*. In this stage, we will simulate a one-soliton case within the interval  $-40 \leq x \leq 40$ ,  $0 \leq t \leq 15$  and taking  $\Delta t = 0,05$  and  $\Delta x = 0,2$  by our hypothetical values. The initial conditions used in this simulation are as follows:

$$u(x, 0) = 3V \operatorname{sech}(x - 5)^2 \quad (3.9)$$

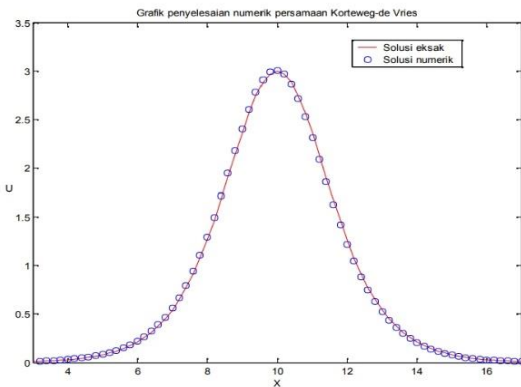


Figure 3.1 Numerical solution of a one-soliton case with  $V=1$

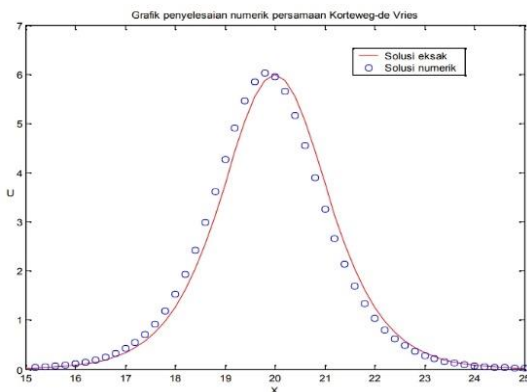


Figure 3.2 Numerical solution of soliton wave with  $V = 2$

The figure 3.1 above shows the numerical solution of the KdV equation at  $t = 15$ , where in equation (3.8), the value of  $V = 1$  with  $\Delta t = 0,05$  and  $\Delta x = 0,2$ . The graph above also shows the comparison between the exact solution and the numerical solution. Based on the graph, the numerical error in the  $t$  interval,  $[0,15]$  for  $\Delta x = 0,2$  and  $V = 1$  is  $1,60 \times 10^{-2}$ . The graph 3.2 shows the numerical solution of the KdV equation at  $t = 15$  with the values  $V = 2$ ,  $\Delta t = 0,05$ , and  $\Delta x = 0,2$ , as given in equation (3.8). The graph also illustrates the comparison between the exact solution and the numerical solution. Based on the graph, the numerical error in the  $t$  interval  $[0,15]$  for  $\Delta x = 0,2$  and  $V=2$  is  $8,6 \times 10^{-2}$ .

By comparing the exact solution with the numerical solution, the maximum numerical error within the interval  $[0,15]$  for  $V = 1$  with  $\Delta t = 0,05$  and  $\Delta x = 0,2$  is  $1,60 \times 10^{-2}$ . Meanwhile, for  $\Delta t = 0,06$  and  $\Delta x = 0,1$ , the maximum numerical error within the interval  $[0,15]$  is  $4,060 \times 10^{-3}$ . This shows that by reducing the values of  $\Delta t$  and  $\Delta x$ , a smaller maximum numerical error within the interval  $[0,15]$  is obtained.

## 4. Conclusion and Advice

### 4.1. Conclusion

Based on the discussion above, the following conclusions can be drawn:

1. The numerical solution of the Korteweg-de Vries (KdV) equation using the finite difference method with the Crank-Nicolson scheme is achieved by discretizing the general equation of KdV. The equation is then linearized to obtain a linear equation from the discretization process.
2. The stability analysis influences the determination of the values of  $\Delta t$  and  $\Delta x$  to be used. Based on the analysis, it is found that the finite difference method with the Crank-Nicolson scheme for the Korteweg-de Vries (KdV) equation is an unconditionally stable method, allowing for the freedom to choose values of  $\Delta t$  and  $\Delta x$ .
3. Based on the comparison between the exact solution and the numerical solution obtained, it can be concluded that the larger the values of  $V$ ,  $\Delta t$ , and  $\Delta x$ , the greater the numerical errors produced.

### 4.2. Advice

Solving the Korteweg-de Vries (KdV) equation has been the subject of numerous studies by various researchers. Therefore, it is advisable to compare each numerical solution for further investigation. Through these comparisons, we can discern the strengths and weaknesses of each method and determine which numerical method is the most suitable for solving the equation.

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