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Stability Analysis of a Predator-prey Model with Anti-Predator Behavior and Allee Effect on Prey

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ABSTRACT

We explore a predator-prey model that incorporates both anti-predator behavior by the prey and the Allee effect, where populationgrowth declines at low densities. Four equilibrium points emerge: extinction for both species (E_0), two predator extinction points (E_1 and E_2), and one coexistence point for both populations (E_3). While the stability of $E_0, E_2,$ and E_3 depends on the given parameters, E_1 is always unstable. We then verified this analysis through numerical simulations using Runge-Kutta method in Python.

Keywords: Predator-Prey, Allee Effect, Anti Predator, Runge-Kutta

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1. Introduction

The relationship between prey and predator is one of the most extensively studied topics in mathematical modeling. The first model that described the interaction between prey and predator, subsequently referred to as the predator-prey model, was introduced by Lotka and Volterra and later became known as the Lotka-Volterra model [1]. Leslie and Gower assumed that the populations of prey and predators grow logistically and are limited by environmental capacity or carrying capacity [2]. The environmental capacity of the predator population is affected by the prey population, while the prey population is limited by a fixed environmental carrying capacity. Subsequently, various models have been modified by altering the response functions, including Holling types 1, 2, 3, and 4 [2, 3, 4, 5].

Besides altering its response functions, the predator-prey model is also adapted by incorporating additional factors such as the Allee effect [3, 5, 9], and anti-predator behavior [4, 6, 8]. The Allee effect represents a phenomenon illustrating how the growth or reproduction of individuals within a population can be impeded or even halted when the population density drops below a certain threshold known as the Allee threshold. The emergence of the Allee effect in the prey population results in individuals struggling to find mates, making reproduction more challenging, consequently leading to a decline in the prey population. Ye et al. [3] explored a model

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incorporating a strong Allee effect and a type 1 response function, which is defined by:

$$\frac{dx}{dt} = rx\left(1-\frac{x}{K}\right)(x-L) - axy; \quad (1)$$
$$\frac{dy}{dt} = c(ax)y - my.$$

x and y represent the populations of prey and predators, respectively. The parameter r describes the intrinsic growth rate of the prey population, and c is the conversion coefficient from prey to predator. The environmental carrying capacity is symbolized by K, while the Allee effect is represented by L. The predation rate by predators is denoted by a, and the predator death rate is symbolized by m.

Gaib and Wahdania [8] investigated a predator-prey model with the Monod-Haldane response function and anti-predator behavior, as follows

$$\frac{dx}{dt} = rx\left(1-\frac{x}{K}\right) - \frac{axy}{x+ny^2}; \quad (2)$$
$$\frac{dy}{dt} = \frac{bxy}{x+ny^2} - my - \eta xy.$$

b is the interaction coefficient between prey and predator influence the growth rate of predators. n represents the saturation level of predation, and anti-predator behavior is symbolized by η. Anti-predator behavior is a mechanism that evolves as an adaptation in prey to enhance their survival. The addition of this factor allows prey to defend themselves against predator attacks and influences the dynamics of both prey and predator populations.

2. Mathematical Model

Based on the introduction presented above, we know that predator-prey interactions are not only influenced by the Allee effect but also anti-predator behavior. Therefore, we modified model (1) by adding anti-predator behavior, as in model (2), so that the model becomes:

$$\frac{dx}{dt} = rx\left(1-\frac{x}{K}\right)(x-L) - axy; \quad (3)$$
$$\frac{dy}{dt} = c(ax)y - my - \eta xy.$$

3. Results and Discussions

3.1. Equilibrium Points

The equilibrium points of model (3) are obtained by solving the equations $\frac{dx}{dt} = 0$ and $\frac{dy}{dt} = 0$. There are four possible fixed points:

1. Extinction of both species point
$$E_0 = (0,0)$$
2. Predator extinction point
$$E_1 = (L,0)$$
3. Predator extinction point
$$E_2 = (K,0)$$
4. Interior equilibrium point where both species coexist:
$$E_3 = (x^*, y^*)$$

where $x^* = \frac{m}{ac-\eta}$;

$$y^* = \frac{r}{a}\left(1-\frac{m}{K(ac-\eta)}\right)\left(\frac{m}{ac-\eta} - L\right).$$

E_3 exist when x^* and y^* positive. x^* will be positive if $ac > \eta$ and y^*will be positive if $\frac{m}{K(ac-\eta)} < 1$ and $L < \frac{m}{ac-\eta}$.

3.2. Stability Analysis of Equilibrium Points

The stability analysis of equilibrium points is conducted by linearizing model (3). The Jacobian matrix of the model is:

$$J(x,y) = \begin{bmatrix} j_{11} & -ax \\ acy - \eta y & acx - \eta x - m \end{bmatrix},$$

where

$$j_{11} = r\left(1-\frac{x}{K}\right)(x-L) - \frac{rx(x-L)}{K} + rx\left(1-\frac{x}{K}\right) - ay.$$

The stability of each equilibrium point can be determined by substituting each equilibrium point into the Jacobian matrix and finding its eigenvalues.

1. $E_0 = (0,0)$

$$J(0,0) = \begin{bmatrix} -rL & 0 \\ 0 & -m \end{bmatrix}.$$

The eigenvalues are $\lambda_1 = -rL$; $\lambda_2 = -m$. Since all parameters have positive values, the eigenvalues of the point E_0 are negative. Therefore, the point E_0 is locally stable.

2. $E_1 = (L, 0)$

$$J(L,0) = \begin{bmatrix} rL\left(1-\frac{L}{K}\right) & -aL \\ 0 & Lac - L\eta - m \end{bmatrix}.$$

The eigenvalues are $\lambda_1 = \frac{rL(K-L)}{K}$; $\lambda_2 = Lac - L\eta - m$. λ_1 and λ_2 will be negative if $L > K$ and $L < \frac{m}{ac-\eta}$. Since L represents the Allee effect and K is the environmental carrying capacity, L cannot be greater than K. Therefore, point E_1 is always unstable.

3. $E_2 = (K, 0)$

$$J(K,0) = \begin{bmatrix} -r(K-L) & -aK \\ 0 & Kac - K\eta - m \end{bmatrix}.$$

The eigenvalues are $\lambda_1 = rL - rK; \lambda_2 = Kac - K\eta - m$. Point E_2 will be stable if both eigenvalues are negative, i.e., when $L < K$ and $K < \frac{m}{ac-\eta}$.

4. $E_3 = (x^*, y^*)$

The Jacobian matrix of $E_3(x^*, y^*)$ is

$$J(x^*,y^*) = \begin{bmatrix} r\left(1-\frac{x^*}{K}\right)(x^*-L) - \frac{rx^*(x^*-L)}{K} + rx^*\left(1-\frac{x^*}{K}\right) - ay^* & -ax^* \\ acy^* - \eta y^* & acx^* - \eta x^* - m \end{bmatrix}.$$

Because it is not straightforward to determine the eigenvalues of the equilibrium point E_3, its stability criteria can be investigated by utilizing trace and determinant values. Point E_3 is stable if the following two statements are satisfied.

a. $Tr\big(J(E_3)\big) < 0$

b. $Det\big(J(E_3)\big) > 0$

The first condition, the trace value of point E_3 is as follows:

$$Tr\big(J(E_3)\big) = r\left(1-\frac{x^*}{K}\right)(x^*-L) - \frac{rx^*(x^*-L)}{K} + rx^*\left(1-\frac{x^*}{K}\right) - ay^* + acx^* - \eta x^* - m$$
$$= rx^* - \frac{rx^{*2}}{K} - rL + \frac{rLx^*}{K} - \frac{rx^{*2}}{K} + \frac{rLx^*}{K} + rx^* - \frac{rx^{*2}}{K} - ay^* + acx^* - \eta x^* - m$$

$$= 2rx^* + \frac{2rLx^*}{K} + acx^* - \left(\frac{3rx^{*2}}{K} + \eta x^* + rL + m + ay^*\right)$$

$$= \left(2r + \frac{2rL}{K} + ac\right)x^* - \left(\frac{3rx^{*2}}{K} + \eta x^* + ay^* + m + rL\right)$$

The value of $Tr\big(J(E_3)\big)$ will be negative if $\left(2r + \frac{2rL}{K} + ac\right)x^* < \frac{3rx^{*2}}{K} + \eta x^* + ay^* + rL + m.$

The second condition, the determinant of point E_3 is as follows

$$Det\big(J(E_3)\big) = \left(r\left(1-\frac{x^*}{K}\right)(x^*-L) - \frac{rx^*(x^*-L)}{K} + rx^*\left(1-\frac{x^*}{K}\right) - ay^*\right)(acx^* - \eta x^* - m) - (-ax^*)(acy^* - \eta y^*)$$

$$= \left(\left(r - \frac{rx^*}{K}\right)(x^*-L) - \frac{rx^{*2}}{K} + \frac{rLx^*}{K} + rx^* - \frac{rx^{*2}}{K} - ay^*\right)(acx^* - \eta x^* - m) + acx^*y^* - a\eta x^*y^*$$

$$= \left(rx^* - \frac{rx^{*2}}{K} - rL + \frac{rLx^*}{K} - \frac{rx^{*2}}{K} + \frac{rLx^*}{K} + rx^* - \frac{rx^{*2}}{K} - ay^*\right)(acx^* - \eta x^* - m) + acx^*y^* - a\eta x^*y^*$$

$$= \left(2rx^* + 2rLx^* - \frac{3rx^{*2}}{K} - rL - ay^*\right)(acx^* - \eta x^* - m) + acx^*y^* - a\eta x^*y^*$$

$$= 2acrx^{*2} + 2acrLx^{*2} - \frac{3acrx^{*3}}{K} - acrLx^* - a^2cx^*y^* - 2\eta rx^{*2} - 2\eta rLx^{*2} + \frac{3\eta rx^{*3}}{K} + \eta rLx^* + a\eta x^*y^* - 2mrx^* - 2mrLx^* + \frac{3mrx^{*2}}{K} + mrL + amy^* + acx^*y^* - a\eta x^*y^*$$

$$= 2acrx^{*2} + 2acrLx^{*2} + \frac{3\eta rx^{*3}}{K} + \eta rLx^* + a\eta x^*y^* + \frac{3mrx^{*2}}{K} + mrL + amy^* + acx^*y^* - \frac{3acrx^{*3}}{K} - acrLx^* - a^2cx^*y^* - 2\eta rx^{*2} - 2\eta rLx^{*2} - 2mrx^* - 2mrLx^* - a\eta x^*y^*$$

$$= \frac{3\eta rx^{*3}}{K} + \left(\frac{3mr}{K} + 2acrL + 2acr\right)x^{*2} + \eta rLx^* + (a\eta + ac)x^*y^* + mrL + amy^* - \left(\frac{3acr}{K}x^{*3} + 2\eta rx^{*2}(1+L) + (acrL + 2mrL + 2mr)x^* + (a^2c + a\eta)x^*y^*\right).$$

Let

$$\omega_1 = \frac{3\eta rx^{*3}}{K} + \left(\frac{3mr}{K} + 2acrL + 2acr\right)x^{*2} + \eta rLx^* + (a\eta + ac)x^*y^* + mrL + amy^*;$$

$$\omega_2 = \frac{3acr}{K}x^{*3} + 2\eta rx^{*2}(1+L) + (acrL + 2mrL + 2mr)x^* + (a^2c + a\eta)x^*y^*.$$

So that $Det\big(J(E_3)\big) = \omega_1 - \omega_2$. $Det\big(J(E_3)\big)$ will be positive if $\omega_1 > \omega_2$.

4. Numeric Simulations

After conducting the stability analysis of the predator-prey model with anti-predator behavior and Allee effect on prey, this section involves numeric simulation using various parameters to visually demonstrate the system's behavior. For the numeric simulation, the fourth-order Runge-Kutta method is employed, utilizing Python programming.

4.1. Simulation 1

The parameter values used are $r = 0.8, K = 30, L = 2, a = 0.6, c = 0.8, m = 0.3, \eta = 0.2$ and the initial conditions are $x_0 = 8$ and $y_0 = 1.5$.

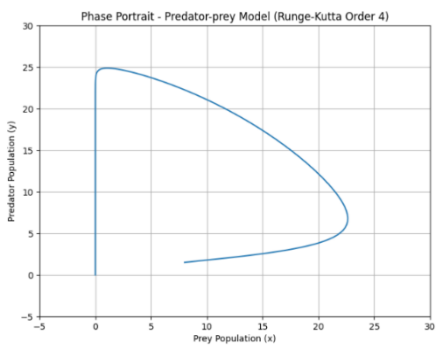

Figure. 1 –The solution moves towards point E_0

Based on the simulation results using the given parameters and initial values, it is evident that the solution converges towards point E_0. The prey population gradually increases, followed by a significant rise in the predator population. Over time, both populations decline and eventually approach the extinction points for both predator and prey.

4.2. Simulation 2

In this simulation, the parameters used are $r = 0.8, K = 20, L = 0.3, a = 1, c = 0.5, m = 0.5, \eta = 0.5$. The initial values remain the same, namely $x_0 = 8$ and $y_0 = 1.5$.

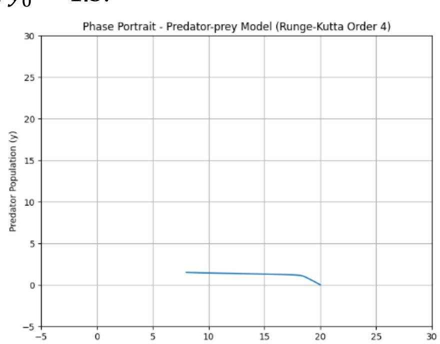

Figure. 2 – The solution moves towards point E_2

With the given parameters, the system's solution converges towards point E_2. This implies predator extinction, consequently causing the prey population to increase towards the environmental carrying capacity, K.

4.3. Simulation 3

In this simulation, the parameters used are $r = 0.3, K = 30, L = 0.02, a = 0.6, c = 0.36, m = 0.3$ and $\eta = 0.2$. The initial values remain the same, namely $x_0 = 8$ and $y_0 = 1.5$.

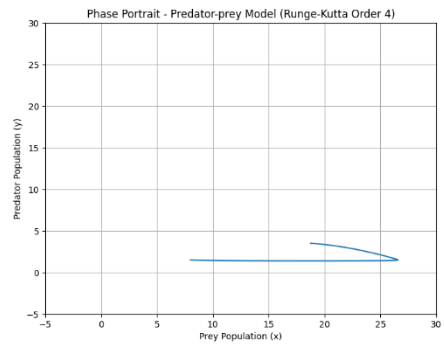

Figure. 3 – The solution moves towards point E_3

In this simulation, the solution moves towards the interior equilibrium point E_3, which represents the coexistence point of the two species.

5. Conclusion

Analysis of model (3) reveals four equilibrium points, namely, E_0, E_1, E_2, and E_3. Point E_0 is locally stable, while E_1 is always unstable. Since L represents Allee effect and K is the environmental carrying capacity, L cannot be greater than K, making point E_1 consistently unstable. Point E_2 will be stable if $L < K$ and $K < \frac{m}{ac-\eta}$, and the stability of interior point E_3, or coexistence of the two species, depends on the selected parameters. From the simulation it is observed that suppressing the Allee effect and anti-predator behavior fosters stability at E_3, hinting that minimal ecological pressures favor coexistence.

REFERENCES

[1] M. Putra, A. J. Dwi dan A. I. A Himayati. *Stability Analysis of Leslie-Gower Model with Herd Behavior on Prey. Indonesia Journal of Pure and Applied Mathematics*. 2022. Vol 4. No. 1, 65-71.

[2] M. Putra, A.J. Dwi, A. Suryanto dan N. Shofianah. Stability analysis and Hopf bifurcation of intraguild predation model with Holling type II functional response. Int. J. Ecol. Dev. 32 (2017), no. 4, 110–118.

[3] Ye, Y. , Hua Liu, Yumei Wei, Kai Zhang, MingMa dan Jianhua Ye. Dynamic study of a predator-prey model with Allee effect and Holling type-I functional response. Advances in Difference Equations. 2019. 2019-369.

[4] Saadah, S., Abadi dan Dian Savitri. Model Interaksi Mangsa Pemangsa dengan Fungsi Respon Rasio Dependent Holling Tipe II dan Perilaku Anti Pemangsa. Jurnal Ilmiah Matematika. 2019. Vol 7. No. 2.

[5] Indrajaya, D.; Suryanto, A.; Alghofari, A.R. Dynamics of modified Leslie-Gower predator-prey model with Beddington-DeAngelis functional response and additive Allee effect. Int. J. Ecol. Dev. 2016, 31, 60–71.

[6] Yang, R., Xiao Zhao dan Yong An. Dynamical Analysis of a Delayed Diffusive Predator–Prey Model with Additional Food Provided and Anti-Predator Behavior. Mathematics. 2022, 10, 469.

[8] Gaib, M. B. dan Wahdania At. Ja'a. Analisis Kestabilan Model Interaksi Predator-prey dengan Fungsi Respon Monod-Haldane dan Perilaku Anti Predator. Euler: Jurnal Ilmiah Matematika, Sains dan Teknologi. 2020. Vol. 8, No. 2, 51-59.

[9] Suryanto, A. Isnani Darti, dan Syaiful Anam. Stability Analysis of a Fractional Order Modified Leslie-Gower Model with Additive Allee Effect. International Journal of Mathematics and Mathematical Sciences. Vol. 2017.