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Solution of The Duffing Equation Using Exponential Time Differencing Method

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ABSTRACT

To describe the spring stiffening effect that occurs in physics and engineering problems, Georg Duffing added the cubic stiffness term to the linear harmonic oscillator equation and is now known as the Duffing oscillator. Despite its simplicity, its dynamic behavior is very diverse. In this research, the Exponential Time Difference (ETD) method is introduced to solve the Duffing oscillator numerically. To formulate the ETD method, we were using the integration factors. It is a function which, when multiplied by an ordinary differential equation, produces a differential equation that can be integrated. This method is an effective numerical method for solving complex differential equations, especially equations that have strong non-linearity, including the Duffing oscillator. The ETD method delivers highly accurate numerical solutions for the Duffing oscillator, with minimal discrepancy from the analytical results. Through parameter variation, the ETD method's applicability extends to diverse Duffing oscillator configurations.

Keywords: Duffing Oscillator; Exponential Time Difference; Integration Factors; non-linear.

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1. Introduction

To describe the spring hardening effect that occurs in physics and engineering problems, Georg Duffing added a cubic stiffness term to the linear harmonic oscillator equation and it is now known as the Duffing oscillator oscillator (Duffing, 1918). In a classical mechanical system, this model can be described by a mass on a spring that has stiffness in it. The differential equation for this model is:

$$\frac{d^2x}{dt^2} + \delta\frac{dx}{dt} + \alpha x + \beta x^3 = \gamma\cos(\omega t) \tag{1}$$

where x is the displacement and $\delta, \alpha, \beta, \gamma, \omega$ are constant parameters related to damping, stiffness, restoring force, driving force amplitude, and driving force angular frequency, respectively.

The Duffing oscillator is used as an approximation model for physical systems or as a model to test various solutions of new methods (C. Pezeshki, 1987; Feng, 2003). The Duffing oscillator has been solved in various ways, such as the P-stable linear symmetric multistep method (Wang, 2005), Laplace decomposition algorithm (Yusufoğlu, 2006), and target function method (Chen, 2002).

Because of its simplicity, and because so much is already known about the Duffing equation, it is used by many researchers as an approximation model of many physical systems or as a convenient mathematical model to investigate new solution methods (Johannessen, 2014; Johannessen, 2015).

On the other hand, the Duffing differential equation has also been effectively addressed in many studies. Despite the simplicity of the Duffing oscillator, its dynamic behavior is

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very diverse and its research is still ongoing today. In this study, the Exponential Time Differencing (ETD) method is introduced to numerically solve the Duffing oscillator.

2. Research Methods

To formulate the ETD method (S.M. Cox, 2002), it is necessary to explain the integration factor. An integration factor is a function that, when multiplied by an ordinary differential equation, produces an integrable differential equation.

For example, here is a first-order differential equation:

$$\frac{du(t)}{dt} - c\,u(t) = F(u,t) \qquad (2)$$

where $F(u,t)$ corresponds to forcing or non-linear terms, and c is a constant. We start the derivation of the ETD method by introducing the integrating factor e^{-ct} into the derivative (Gregory Beylkin, 1998).

$$\frac{d}{dt}[e^{-ct}\,u(t)] = -c\,e^{-ct}\,u(t) + e^{-ct}\frac{du(t)}{dt}$$
$$\frac{d}{dt}[e^{-ct}\,u(t)] = e^{-ct}\left[-c\,u(t) + \frac{du(t)}{dt}\right]$$
$$\frac{d}{dt}[e^{-ct}\,u(t)] = e^{-ct}\,F(u,t) \qquad (3)$$

If Equation (3) is integrated with time step $h = t_{n+1} - t_n$, then

$$\int_0^h d(e^{-ct}\,u(t)) = \int_0^h e^{-ct}\,F(u,t)\,dt \qquad (4)$$

where the solution of the integral form in Equation (4) is (Adams, 2003)

$$u(h) = e^{ch}\,u(0) + e^{ch}\int_0^h e^{-ct}\,F(u,t)\,dt \qquad (5)$$

Assuming $t_n = 0$, Equation (5) can be written as:

$$u(t_{n+1}) = e^{ch}\,u(t_n) + e^{ch}\int_0^h e^{-c\tau}\,F(u_{t_n},t_n)\,d\tau \qquad (6)$$

If we write $u(t_n) = u_n$ and $F(u_{t_n},t_n) = F_n$, then

$$u_{n+1} = u_n\,e^{ch} + e^{ch}\int_0^h e^{-c\tau}\,F_n\,d\tau$$
$$u_{n+1} = u_n\,e^{ch} + e^{ch}\,F_n\left[-\frac{e^{-c\tau}}{c}\right]_0^h$$
$$u_{n+1} = u_n\,e^{ch} + F_n\left(\frac{e^{ch}-1}{c}\right) \qquad (7)$$

Equation (7) is the ETD method to be used.

3. Results and Discussion

Let us consider a damped Duffing oscillator, suppose it has the form of:

$$\frac{d^2x}{dt^2} + \frac{dx}{dt} + x + x^3 = cos^3(t) - \sin(t) \qquad (8)$$

with initial conditions are

$$x(0) = 1 \quad \text{and} \quad x'(0) = 0 \qquad (9)$$

The exact solution is (Tabatabaei, 2014)

$$x(t) = \cos(t) \qquad (10)$$

If we write

$$\frac{dx}{dt} = v \quad \text{dan} \quad \frac{d^2x}{dt^2} = \frac{dv}{dt} \qquad (11)$$

then, Equation (8) can be arranged in such a way that it takes the form of

$$\frac{dv}{dt} + v = -x - x^3 + cos^3(t) - \sin(t) \qquad (12)$$

We can see that Equation (12) is similar to Equation (2) if

$$c = -1 \quad \text{and} \quad F(x,t) = -x - x^3 + cos^3(t) - \sin(t) \qquad (13)$$

Once we have the values of c and $F(x,t)$, we can use the ETD method by iterating on Equation (7). The iteration results can be seen in Figure 1.

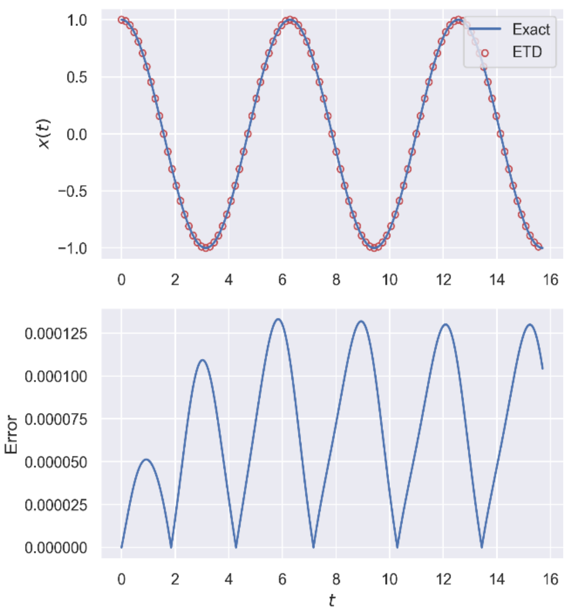

Figure. 1 - Comparison of the solution to Equation (8) using the ETD method with its analytical solution (upper) and discrepancies between them (lower).

In Figure 1, a comparison between the numerical solution given by the ETD method and the analytical solution is shown. The numerical solution given by the ETD method follows well the movement of the analytical solution, with very small discrepancies.

The simulation results show that the ETD method can be used to solve the Duffing oscillator, Equation (8), numerically with good accuracy. It is shown from the largest error obtained is very small, which is 0.0001. The error obtained is calculated by taking the absolute value of the difference between the numerical solution and the analytical solution.

For a different formulation of the Duffing oscillator than the one mentioned above, we can manipulate the equation in such a way that it will yield different values of c and $F(x,t)$ in Equation (13). Therefore, by adjusting these values, we can use the ETD method to solve various differential forms of the Duffing oscillator in a relatively fast and accurate manner.

4. Conclusion

The ETD method has been used to numerically solve the Duffing oscillator model equation. It is one of the effective numerical methods for solving complex differential equations, especially those with strong non-linearity.

The simulation results show that the solution provided by the ETD method and the analytical solution are highly accurate. The difference between the two solutions is very small, even for long time spans. This shows that the ETD method can be used to solve the Duffing oscillator model equation with high accuracy.

Overall, the ETD method is an effective and efficient numerical method for solving the Duffing oscillator model equations. This method can be used for various purposes related to the Duffing oscillator, such as simulation, analysis, and design.

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