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Algebraic Structures and Combinatorial Properties of Unit Graphs in Rings of Integer Modulo with Specific Orders

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ABSTRACT

Unit graph is the intersection of graph theory and algebraic structure, which can be seen from the unit graph representing the ring modulo n in graph form. Let R be a ring with nonzero identity. The unit graph of R, denoted by $G(R)$, has its set of vertices equal to the set of all elements of R; distinct vertices x and y are adjacent if and only if $x + y$ is a unit of R. In this study, the unit graph, which is in the ring of integers modulo n, denoted by $G(\mathbb{Z}_n)$. It turns out when n is 2^k, $G(\mathbb{Z}_n)$ forms a complete bipartite graph for $k \in \mathbb{N}$, whereas when n is prime, $G(\mathbb{Z}_n)$ forms a complete $(n+1)/2$-partites graph. Additionally, the numerical invariants of the graph $G(\mathbb{Z}_n)$, such as degree, chromatic number, clique number, radius, diameter, domination number, and independence number complement the characteristics of $G(\mathbb{Z}_n)$ for further research.

Keywords: numerical invariant, unit graph, ring of integer modulo

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1. Introduction

The unit graph is a special type of graph used to represent an algebraic structure. The vertices of the unit graph represent all the elements of a ring, while the edge of the graph connects two vertices if and only its addition is a unit of the ring. (Pirzada &Altaf, 2022). A unit in a ring R is an element $u \in R$ that has a multiplicative inverse $v \in R$ meaning $u \cdot v = v \cdot u = 1$, where 1 is the identity element for multiplication in R. The set of all such units in R forms a group under the operation of multiplication, usually represented as $U(R)$. Studying unit graph of $\mathbb{Z}_n$ reveals important insights into the nature of the ring's units, such as their distribution, the graph's connectivity, and the numerical invariant.

This article is written to investigate the characteristics of unit graph of the ring of integers modulo n. The study of this graph also contributes to the broader field of algebraic graph theory and its applications. One of them is being able to provide important information for the search for topological indices that are widely applied in the field of chemistry (Malik at.all,2024).

2. Result and Discussion

To understand the connection between algebraic structure and graph theory, Ashrafi, et al. introduce the concept of unit graphs. These graphs illustrate the interactions between ring elements in a visual format. Herein, we shall establish a formal definition of a unit graph.

Definition 1. The unit graph of R, denoted by $G(R)$, has its set of vertices equal to the set of all elements of R; distinct vertices x and y are adjacent if and only if $x + y$ is a unit of R (Ashrafi, et al., 2010).

with the concept of unit graphs established, now we examine their structure for integer modulo rings $\mathbb{Z}_n = \{0, 1, 2, \dots, n-1\}$ under operations, addition and multiplication.

Example 1. Graph unit of $\mathbb{Z}_n$ for $n = 4$.

$$\mathbb{Z}_4 = \{0,1,2,3\}$$

The set of units $\mathbb{Z}_4$, $U(\mathbb{Z}_4) = \{1,3\}$. Then, the vertices that adjacent are $\{(0,1), (0,3), (1,2), (2,3)\}$.

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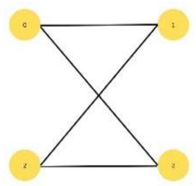

Picture 1. Graph unit of $\mathbb{Z}_4$

Theorem 1. Let $\mathbb{Z}_n$ be a ring with order $n = 2^k$ for $k \in \mathbb{N}$. Then, unit graph $G(\mathbb{Z}_n)$ is a complete bipartite graph.

Proof. To prove the graph is a bipartite graph, we divide the vertices of $G(\mathbb{Z}_{2^k})$ into two partitions, V_1 and V_2, such as

$$V_1 = \{1, 3, 5, 7, \ldots, 2^{k-1} + 1\}$$
$$V_2 = \{0, 2, 4, 6, \ldots, 2^{k-1}\}$$

Note that V_1 is containing all of odd integers modulo 2^k means that for each element v of V_1 there exists an element u of $\mathbb{Z}_{2^k}$, such that uv is equivalent to $1(\,mod 2^k)$, so V_1 is the unit set of $\mathbb{Z}_{2^k}$. Because, V_1 is the set of odd integers modulo 2^k, $\forall x, y \in V_1$, then $x + y$ is even, hence $x + y \in V_2$ in other word $x + y$ not a unit, consequently x and y are not adjacent.

It is easy to check that V_2 is containing all non-unit of ring $\mathbb{Z}_{2^k}$, Since every element of V_2 are even then, $x + y \in V_2$, then $x + y$ not a unit, hence x and y are not adjacent.

Lastly, for any $x \in V_1$ and $y \in V_2$ we have $x + y$ must be odd, then $x + y \in V_1$, hence x and y are adjacent. Therefore, every vertex of V_1 are adjacent to every vertex of V_2. Then, $G(\mathbb{Z}_{2^k})$ is a complete bipartite graph ■

We now turn our consideration to another interesting case: if the order is a prime number.

Theorem 2. Let $\mathbb{Z}_n$ be an integer modulo ring. Then, the unit graph $G(\mathbb{Z}_n)$ is a complete $\frac{n+1}{2}$ partite graph for n an odd prime number.

Proof. Let $\boldsymbol{n}$ be an odd prime number, then we can divide $\mathbb{Z}_n$ into $\frac{n+1}{2}$ partitions:

$$V_0 = \{0\}$$
$$V_1 = \{1, n-1\}$$
$$V_2 = \{2, n-2\}$$
$$\vdots$$
$$V_{\frac{n-1}{2}} = \left\{\frac{n-1}{2}, \frac{n+1}{2}\right\}$$

It is clear that if $i + (n - i) = 0$ then i and $n - i$ are not adjacent. And for any $x \in V_i$, $y \in V_j$, $i \neq j$, we have $x + y \neq 0$, then $x + y$ is a unit, hence x and y are adjacent. Then it's proven that the graph unit $G(\mathbb{Z}_n)$ is a complete $\frac{n+1}{2}$ − partite graph. ■

Considering Theorem 1 and 2, we established the structural properties of the unit graph of the ring of integers modulo n, for a prime number. Further, we examined the numerical invariants associated with this graph. These invariants will further elucidate the characteristics and complexity of the unit graphs in these contexts.

Definition 2. The diameter of G, $\text{diam}(G)$, is the maximum distance between vertices of G (Jafari & Musawi, 2024).

The diameter of a graph represents the maximum distance between any pair of vertices, highlighting the graph's overall extent. In contrast, the radius is the minimum eccentricity of its vertices, indicating the smallest maximum distance from any vertex to all others.

Definition 3. The radius of G is the minimum eccentricity among the vertices of G, i.e., $\text{rad}(G) = \min\{\varepsilon(x) | x \in V(G)\}$ (Li & Zu, 2021).

Definition 4. For a graph G the domination number, denoted $\gamma(G)$ is the smallest size of a dominating set for G. A dominating set is a subset D of the vertices $V(G)$ such that every vertex not in D is adjacent to at least one vertex in D (Nurhabibah, et al., 2023).

Definition 5. Let G be a graph, the independence number of G, denoted by $\beta(G)$ is the maximum cardinality of the independence set of G. The independence set is a subset of $V(G)$ such that no two vertices in the subset represent an edge of $V(G)$ (Nurhabibah, et al., 2023).

Definition 6. For a graph G, the chromatic number $\chi(G)$ is defined as the minimum number of colors required to color the vertices of G in such a way that no two adjacent vertices share the same color. (Isaev & Kang, 2021).

Definition 7. The clique number of a graph is the size of the largest clique in the graph, where a clique is a subset of vertices such that every two distinct vertices are adjacent (West, 2001).

The unit graph of ring of integer modulo has several fundamental properties.

Theorem 3. Let $\mathbb{Z}_n$ be an integer modulo ring with order $n = 2^k$ for $k \in \mathbb{N}$. Then, unit graph $G(\mathbb{Z}_n)$ has several numerical invariants:

1. $deg(v) = 2^{k-1}, \forall\, v \in V\left(G(\mathbb{Z}_{2^k})\right)$.
2. The chromatic number of $G(\mathbb{Z}_{2^k})$, $\chi\left(G(\mathbb{Z}_{2^k})\right) = 2$.
3. The clique number of $G(\mathbb{Z}_{2^k})$, $\omega\left(G(\mathbb{Z}_{2^k})\right) = 2$.
4. The diameter of $G(\mathbb{Z}_{2^k})$ is 2.
5. The radius of $G(\mathbb{Z}_{2^k})$ is 1.
6. The domination number of $\gamma\left(G(\mathbb{Z}_{2^k})\right)$ is 2.
7. The independent number $\beta\left(G(\mathbb{Z}_{2^k})\right)$ is 2^{k-1}.

Proof From Theorem 1, it has been proved that $G(\mathbb{Z}_{2^k})$ for $k \in \mathbb{N}$ is a complete bipartite graph. $G(\mathbb{Z}_{2^k})$ has two partitions, V_1 and V_2, such as

$$V_1 = \{1, 3, 5, 7, \ldots, 2^{k-1} + 1\}$$
$$V_2 = \{0, 2, 4, 6, \ldots, 2^{k-1}\}$$

1. It is easy to check that $|V_1| = |V_2| = \frac{2^k}{2} = 2^{k-1}$, because $G(\mathbb{Z}_{2^k})$ is a complete bipartite graph, of course each vertex of V_1 connected to each vertex of V_2. Hence, $deg(v) = 2^{k-1}, \forall\, v \in V\left(G(\mathbb{Z}_{2^k})\right)$.

2. Since $G(\mathbb{Z}_{2^k})$ is a complete bipartite graph, it means $G(\mathbb{Z}_{2^k})$ is a graph whose vertices can be divided into disjoint sets such that no two vertices within the same set are adjacent. But, every vertex of V_1 is adjacent with every vertex of V_2. So, it is clearly that every vertex in the same partition has the same color. Then, the minimal number of colors used is 2. Hence, $\chi\left(G(\mathbb{Z}_{2^k})\right) = 2$.
3. Let $W \subset V$, where every pair of vertices in this subset is connected by an edge. Since $G(\mathbb{Z}_{2^k})$ is a complete bipartite graph, every vertex of V_1 can only be adjacent to every vertex of V_2. So, W forming pair $\{u, v\}$ where $u \in V_1$ and $v \in V_2$. Any attempt to add more vertices to W from V_1 dan V_2 would result not all vertices in $W \subset V$ are adjacent, violating the definition of a clique number. Thus, the largest possibly clique in a complete bipartite graph $G(\mathbb{Z}_{2^k})$ has size 2 or $\omega\left(G(\mathbb{Z}_{2^k})\right) = 2$.
4. Since a complete bipartite graph $G(\mathbb{Z}_{2^k})$ connects every vertex in V_1 to every vertex in V_2 and there are no edges within the same set, therefore Any vertex in V_1 is directly connected to any vertex in V_2, making the distance between them 1 and Any vertex in set V_1 can reach any other vertex in V_1 by going through a vertex in set V_2, making the distance 2 (and vice versa for vertices in set V_2). Therefore, the $diam\left(G(\mathbb{Z}_{2^k})\right) = 2$.
5. In the complete bipartite graph $G(\mathbb{Z}_{2^k})$ every vertex in V_1 is adjacent to every vertex in V_2, and there is no edges within the same set. Consequently, any vertex in V_1 is adjacent to any vertex in V_2, making the distance between them 1. Additionally, any vertex in V_1 can reach another vertex in V_1 by passing through a vertex in V_2, resulting in a distance of 2 (and similarly for vertices in V_2. Since the radius is the minimum distance between any two vertices, the radius of $\left(G(\mathbb{Z}_{2^k})\right)$ is 1.
6. Since $G(\mathbb{Z}_{2^k})$ is complete bipartite graph, such as V_1 is the set of the odd number and V_2 is the set of the even number. Choose any arbitrary $x_i \in V_1$ and $x_j \in V_2$. x_i is adjacent to any vertices in V_2 and x_j is adjacent to any vertices in V_1. $D = \{x_i, x_j\}$ adjacent to any $V(G) - D$. Then, $\gamma\left(G(\mathbb{Z}_{2^k})\right) = 2$.
7. The $G(\mathbb{Z}_{2^k})$ has 2 partitions, V_1 and V_2, and there are no edges within V_1 or within V_2. Then, the independence number of $G(\mathbb{Z}_{2^k})$ is $\max\{|V_1|, |V_2|\}$. As we know $|V_1| = |V_2| = 2^{k-1}$, then the independent number, $\beta\left(G(\mathbb{Z}_{2^k})\right)$ equal to 2^{k-1}.

Now, we focus to another case: if the order is a prime number. In the theorem 4, we determined some numerical invariants of the unit graph of Z_n for any odd prime number n.

Theorem 4. Let $\mathbb{Z}_n$ be an integer modulo group with order n is odd prime. Then, unit graph $G(\mathbb{Z}_n)$ has several numerical invariants:

1. The degree of $G(\mathbb{Z}_n)$. $deg(\{0\}) = n - 1$ and $deg(v) = n - 2$, for all $v \in V(G(\mathbb{Z}_n))$.
2. The chromatic number of $G(\mathbb{Z}_n)$, $\chi(G(\mathbb{Z}_n)) = \frac{n+1}{2}$
3. The clique number of $G(\mathbb{Z}_n)$, $\omega(G(\mathbb{Z}_n)) = \frac{n+1}{2}$.
4. The diameter of $G(\mathbb{Z}_n)$ is 2.
5. The radius of $G(\mathbb{Z}_n)$ is 1.
6. The domination number of $\gamma(G(\mathbb{Z}_n))$ is 1.
7. The independence number of $\beta(G(\mathbb{Z}_n))$ is 2.

Proof As we know that $G(\mathbb{Z}_n)$ is a complete $\frac{n+1}{2}-$ partite graph. Such that, all partitions consist 2 vertices, except $V_0 = \{0\}$.

1. Since graph $G(\mathbb{Z}_n)$ is a complete $\frac{n+1}{2}-$ partite graph. Hence, Degree of the graph $G(\mathbb{Z}_n)$ as follows.
   a. $\{0\}$ connected to every vertex of graph $G(\mathbb{Z}_n)$ except itself, then the $Deg(G(\{0\})) = n - 1$.
   b. $\forall u, v \in \mathbb{Z}_n \backslash \{0\}$, u is adjacent to every vertex except itself and v, where u and v within the same partition. Hence, the $Deg(G(\mathbb{Z}_n)) = n - 2$.
2. As known that $G(\mathbb{Z}_n)$ has $\frac{n+1}{2}$, such that every partition is mutually adjacent. Consequently, minimal colours that used to colouring the graph $G(\mathbb{Z}_n)$ is $\frac{n+1}{2}$.
3. Let $W \subset V$, where every pair of vertices in this subset is connected by an edge. Since $G(\mathbb{Z}_n)$ is a complete $\frac{n+1}{2}-$ partite graph and every vertex of each partition is mutually adjacent but the vertices within the same partition are not adjacent. So, W forming subset that consist a vertex of each partition. Any attempt to add more vertices to W from $V_0, V_1, V_2, \dots, V_{\frac{n+1}{2}}$ would result not all vertices in $W \subset V$ are adjacent, violating the definition of a clique number. Thus, the largest possibly clique in a complete bipartite graph $G(\mathbb{Z}_n)$ has size $\frac{n+1}{2}$ or $\omega(G(\mathbb{Z}_n)) = \frac{n+1}{2}$.
4. Let $u, v \in G(\mathbb{Z}_n)$, if u and v from same partition then both can be connected through $\{0\}$, hence the distance is two. Furthermore if u and v from different partition then both is directly connected, hence the distance is one. Therefore, the $diaG(\mathbb{Z}_n) = 2$.
5. Since a complete $\frac{n+1}{2}-$partite graph $G(\mathbb{Z}_n)$, and with the same argument used to prove the diameter above, therefore, the $radG(\mathbb{Z}_n) = 1$.
6. $G(\mathbb{Z}_n)$ has a partition $V_0 = \{0\}$ that is adjacent to every vertex in $V(G) - V_0$, and V_0 is it minimum cardinality set of $V(G)$. $Hence,$ V_0 is the domination set of $G(\mathbb{Z}_n)$. Then, $\gamma(G(\mathbb{Z}_n)) = 1$.
7. We know that $G(\mathbb{Z}_n)$ has $\frac{n+1}{2}$ partitions. There are no edges within each partition and every partition consists of two vertices, except V_0. So, the independence number $\beta(G(\mathbb{Z}_n)) = 2$.

3. Conclusion

This investigation into the properties of unit graphs in the ring of integers modulo n. The proofs of these theories demonstrate the properties and structures of unit graphs in the cases where $n = 2^k$ and n is an odd prime number. The obtained results show that the graph unit of $\mathbb{Z}_{2^k}$ forms a complete bipartite graph and the graph unit of $\mathbb{Z}_n$ for n an odd prime number forms a complete $\frac{p+1}{2}$ −partite graph. Besides, the numerical invariants add the characteristic of $G(\mathbb{Z}_n)$, as follows:

For $n = 2^k$:

1. The degree of $G(\mathbb{Z}_n)$. $deg(\{0\}) = n - 1$ and $deg(v) = n - 2$, for all $v \in V(G(\mathbb{Z}_n))$.
2. The chromatic number of $G(\mathbb{Z}_n)$, $\chi(G(\mathbb{Z}_n)) = \frac{n+1}{2}$
3. The clique number of $G(\mathbb{Z}_n)$, $\omega(G(\mathbb{Z}_n)) = \frac{n+1}{2}$.
4. The diameter of $G(\mathbb{Z}_n)$ is 2.
5. The radius of $G(\mathbb{Z}_n)$ is 1.
6. The domination number of $\gamma(G(\mathbb{Z}_n))$ is 1.
7. The independence number of $\beta(G(\mathbb{Z}_n))$ is 2.

For n is an odd prime number:

1. The degree of $G(\mathbb{Z}_n)$. $deg(\{0\}) = n - 1$ and $deg(v) = n - 2$, for all $v \in V(G(\mathbb{Z}_n))$.
2. The chromatic number of $G(\mathbb{Z}_n)$, $\chi(G(\mathbb{Z}_n)) = \frac{n+1}{2}$
3. The clique number of $G(\mathbb{Z}_n)$, $\omega(G(\mathbb{Z}_n)) = \frac{n+1}{2}$.
4. The diameter of $G(\mathbb{Z}_n)$ is 2.
5. The radius of $G(\mathbb{Z}_n)$ is 1.
6. The domination number of $\gamma(G(\mathbb{Z}_n))$ is 1.
7. The independence number of $\beta(G(\mathbb{Z}_n))$ is 2.

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