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A Novel Approach to Topological Indices of the Power Graph Associated with the Dihedral Group of a Certain Order

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A B S T R A C T

Power graph of a group G, represented by Γ_G, is a graph where the vertex set consists of the elements of G. Two distinct vertices $a, b \in G$ are connected by an edge if and only if there exists a positive integer m such that $a^m = b$ or $b^m = a$. This study explores the utilization of a new approach to compute the topological indices of power graph associated with dihedral group with $n = p^k$, p is primes and $k \in \mathbb{Z}$. Results obtained indicate that the topological indices calculated using new approach yield the same values as those obtained with the conventional approach.

Keywords: power graph, dihedral group, topological indices, new approach

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1. Introduction

In recent years, research on graphs that used as representations of groups has developed rapidly. Several types of graphs that can be used to represent a group include coprime graph [1], non-coprime graph [2], relative coprime graph [3], intersection graph [4], and power graph [5]. The concept of a power graph of a finite group G (denoted $\Gamma(G)$) was first introduced by Kelarev and Quinn [6]. In their formulation, each element of G is represented as a vertex, and two elements $a, b \in G$ are adjacent whenever there exists a natural number k such that $a^k = b$, which leads to a directed graph. Later, Chakrabarty et al. proposed an undirected version of power graphs for semigroups and cyclic groups by defining two distinct elements a and b to be adjacent if and only if $a^k = b$ or $a = b^k$ for some natural number k [7].

In 2021, Asmarani et al. conducted research on the power graph associated with the dihedral group [5]. This research was extended in 2023 with a focus on the topological indices of power graph [8]. Subsequently, in 2022, Syechah et al. conducted research on the power graph on Z_n [9].

In this study, a new approach to calculating topological indices will be presented. This approach was first introduced by Maulana et al. in 2024, where it was applied to determine the topological indices of zero divisor graphs [10]. In the same year, Syarifudin utilized this approach to determine the topological indices of line graphs on prime ideal graphs [11]. Based on this background, the present research will discuss the application of this new approach in calculating the topological indices for the power graph associated with the dihedral group.

2. Preliminaries

In this part, we provide various definitions, theorems, and lemmas that serve as the foundation for this research. These fundamental concepts are presented to establish a clear and coherent theoretical framework upon which the subsequent analysis is built.

Several definitions of topological indices are presented, including the Wiener index, Gutman index, Harary index, and Hyper-Wiener index.

Definition 2.1. [12] Wiener Index
The Wiener index of Γ written as

$$W(G) = \sum_{u,v \in V(G)} d(u,v)$$

where $d(u,v)$ is the distances of unordered pair of vertex u and v.

Definition 2.2. [13] Hyper-Wiener and Harary index
Let Γ be a graph.

1. The Hyper-Wiener index of Γ, symbolized by $WW(\Gamma)$, written as

$$WW(\Gamma) = \frac{1}{2}\left(\sum_{u,v \in V(\Gamma)} d(u,v) + \sum_{u,v \in V(\Gamma)} d(u,v)^2 \right)$$

   where $d(u,v)$ represents the distance between the unordered pair of vertices u and v.
2. The Harary index of Γ, symbolized by $H(\Gamma)$, written as

$$H(\Gamma) = \sum_{u,v \in V(\Gamma)} \frac{1}{d(u,v)}$$

   where $d(u,v)$ is the distance between the unordered pair of vertices u and v.

Definition 2.3. [14] First Zagreb and Second Zagreb Index
Let Γ be a graph.

1. The First Zagreb Index of Γ, Symbolized by $M_1(\Gamma)$, written as

$$M_1(\Gamma) = \sum_{u \in V(\Gamma)} (d_u)^2$$

   where d_u represents the degree of the vertex u, i.e., the number of edges connected to u.

2. The Second Zagreb Index of Γ, Symbolized by $M_2(\Gamma)$, written as

$$M_2(\Gamma) = \sum_{u,v \in E(\Gamma)} d_u d_v$$

where u and v are the endpoints of an edge in Γ.

Definition 2.4. [15] Gutman index
The Gutman index of Γ, symbolized by $Gut(\Gamma)$, written as

$$Gut(\Gamma) = \sum_{\{u,v\} \subseteq V(\Gamma)} (d_u d_v) d(u, v).$$

Here the new approach of topological indices based on [10].

Lemma 2.5. [10] Consider Γ as a graph with $\text{diam}(\Gamma) \leq 2$. The Wiener index of Γ is defined as $|V(\Gamma)|(|V(\Gamma)| - 1) - |E(\Gamma)|$.

Proof. Since $\text{diam}(\Gamma) \leq 2$, the number of unordered pairs of vertices in Γ that have distance 2 is

$$\binom{|V(\Gamma)|}{2} - |E(\Gamma)|.$$

Hence, the Wiener index of Γ is

$$\begin{aligned} W(\Gamma) &= |E(\Gamma)| + 2\left(\binom{|V(\Gamma)|}{2} - |E(\Gamma)|\right) \\ &= |V(\Gamma)|(|V(\Gamma)| - 1) - |E(\Gamma)|. \end{aligned}$$

□

Lemma 2.6. [10] Consider Γ as a graph with $\text{diam}(\Gamma) \leq 2$. The Hyper-wiener index of Γ is defined as $\frac{3}{2}|V(\Gamma)|(|V(\Gamma)| - 1) - 2|E(\Gamma)|$.

Proof. Since $\text{diam}(\Gamma) \leq 2$, the number of unordered pairs of vertices in Γ is

$$\binom{|V(\Gamma)|}{2} - |E(\Gamma)|.$$

Hence, the hyper-Wiener index of Γ is

$$\begin{aligned} WW(\Gamma) &= \frac{1}{2}\left[|E(\Gamma)| + 2\left(\binom{|V(\Gamma)|}{2} - |E(\Gamma)|\right) + |E(\Gamma)| + 4\left(\binom{|V(\Gamma)|}{2} - |E(\Gamma)|\right)\right] \\ &= \frac{3}{2}|V(\Gamma)|(|V(\Gamma)| - 1) - 2|E(\Gamma)|. \end{aligned}$$

□

Lemma 2.7. [10] Consider Γ as a graph with $\text{diam}(\Gamma) \leq 2$. The Harary index of Γ is defined as $\frac{1}{4}|V(\Gamma)|(|V(\Gamma)| - 1) + \frac{1}{2}|E(\Gamma)|$.

Proof. Since diam(Γ) ≤ 2, the number of unordered pairs of vertices in G that have distance 2 is

$$\binom{|V(\Gamma)|}{2} - |E(\Gamma)|.$$

Hence, the Harary index of Γ is

$$\begin{aligned} H(\Gamma) &= |E(\Gamma)| + \frac{1}{2}\left(\binom{|V(\Gamma)|}{2} - |E(\Gamma)|\right) \\ &= \frac{1}{4}|V(\Gamma)|(|V(\Gamma)| - 1) + \frac{1}{2}|E(\Gamma)|. \end{aligned}$$

□

Lemma 2.8. [10] Consider Γ as a graph with diam(Γ) ≤ 2. The Gutman index of Γ is defined as $Gut(\Gamma) = 4|E(\Gamma)|^2 - M_1(\Gamma) - M_2(\Gamma)$.

Proof. Note that

$$\begin{aligned} 4|E(\Gamma)|^2 &= \left(\sum_{u\in V(\Gamma)} d_u\right)^2 \\ &= \sum_{u\in V(\Gamma)} (d_u)^2 + 2\sum_{uv\in E(\Gamma)} d_u d_v + 2\sum_{uv\notin E(\Gamma)} d_u d_v \\ &= M_1(\Gamma) + 2M_2(\Gamma) + 2\sum_{uv\notin E(\Gamma)} d_u d_v. \end{aligned}$$

Since $diam(\Gamma) \le 2$, then

$$\begin{aligned} Gut(\Gamma) &= \sum_{uv\in E(\Gamma)} (d_u d_v) d(u,v) + \sum_{uv\notin E(\Gamma)} (d_u d_v) d(u,v) \\ &= \sum_{uv\in E(\Gamma)} d_u d_v + 2\sum_{uv\notin E(\Gamma)} d_u d_v \\ &= M_2(\Gamma) + 4|E(\Gamma)|^2 - M_1(\Gamma) - 2M_2(\Gamma) \\ &= 4|E(\Gamma)|^2 - M_1(\Gamma) - M_2(\Gamma). \end{aligned}$$

□

The last two theorem in this section are about degree and topological indices from previous research.

Theorem 2.9. [5] The vertex degree of the power graph of D_{2n}, where $n = p^m$ with p is primes and $m \in \mathbb{Z}$, is given by

$$d_u = \begin{cases} 2n-1 & , u = e \\ n-1 & , u \in \{a, a^2, \ldots, a^{n-1}\} \\ 1 & , u \in \{b, ab, a^2b, \ldots, a^{n-1}b\} \end{cases}.$$

Theorem 2.10. [8] Let $\Gamma_{D_{2n}}$ is power graph of D_{2n}. If $n = p^k$ with p is primes and $k \in \mathbb{Z}$. Then the first Zagreb Index, Wiener Index and Gutman Index of $\Gamma_{D_{2n}}$ are

1. $M_1(\Gamma_{D_{2n}}) = n^2(n+1)$,
2. $W(\Gamma_{D_{2n}}) = \dfrac{7n^2 - 5n}{2}$,
3. $Gut(\Gamma_{D_{2n}}) = \dfrac{(n^4 + n) + 3(n^3 - n^2)}{2}$.

The previous researcher proved Theorem 2.10 with Definition 2.1, Definition 2.3, and Definition 2.4.

Theorem 2.11. [16] Let $\Gamma_{D_{2n}}$ is power graph of D_{2n}. If $n = p^k$, p is primes and $k \in \mathbb{Z}$. Then Second Zagreb Index of $\Gamma_{D_{2n}}$ is

$$M_2(\Gamma_{D_{2n}}) = \frac{n^4 - n^3 + 3n^2 - n}{2}.$$

3. Result

In this section, we first present the definition of the undirected power graph, followed by a new approach for computing the Wiener index, Hyper-Wiener index, Harary index, and Gutman index.

Definition 3.1. [7] Suppose G is a group and $a, b \in G$. The power graph of G (denoted $\Gamma(G)$) is a graph consisting of the set of vertices which are the elements of the group, and the vertices $a, b \in G$ are mutually adjacent if and only if there exists a natural number n such that $a^n = b$ or $a = b^n$.

Theorem 3.2. [5] Let $\Gamma_{D_{2n}}$ is power graph of D_{2n}. If $n = p^k$, p is primes and $k \in \mathbb{Z}$. Then diameter of $\Gamma_{D_{2n}}$ is 2.

Based on Theorem 3.2 then we can use the new approach topological indices for Lemma 2.5 to Lemma 2.8. To use the new approach, we need to find the edges of $\Gamma_{D_{2n}}$.

Theorem 3.3. Edges
Let $\Gamma_{D_{2n}}$ is power graph of D_{2n}. If $n = p^k$, p is primes and $k \in \mathbb{Z}$. Then the total edges of $\Gamma_{D_{2n}}$ is

$$\left|E(\Gamma_{D_{2n}})\right| = \frac{n^2 + n}{2}.$$

Proof. To prove this theorem, we use handshaking lemma and Theorem 2.9. The result is as follows

$$\begin{aligned}
2\left|E(\Gamma_{D_{2n}})\right| &= \sum_{u \in D_{2n}} d_u \\
&= \sum_{u=e} d_e + \sum_{u \in \langle a \rangle} d_u + \sum_{u \in \{b, \ldots, a^{n-1}b\}} d_u \\
&= 2n - 1 + (n-1)(n-1) + n \cdot 1 \\
&= 3n - 1 + n^2 - 2n + 1 \\
2\left|E(\Gamma_{D_{2n}})\right| &= n^2 + n \\
\left|E(\Gamma_{D_{2n}})\right| &= \frac{n^2 + n}{2}.
\end{aligned}$$

□

The three theorems below discuss topological indices related to distance use new novel approach or new point of view of the proof process.

Theorem 3.4. Wiener Index
Let $\Gamma_{D_{2n}}$ is power graph of D_{2n}. If $n = p^k$, p is primes and $k \in \mathbb{Z}$. Then the Wiener Index of $\Gamma_{D_{2n}}$ is

$$W(\Gamma_{D_{2n}}) = \frac{7n^2 - 5n}{2}.$$

Proof. Let $\Gamma_{D_{2n}}$ then $|V(\Gamma_{D_{2n}})| = 2n$. Based on Theorem 3.3 and Lemma 2.5 we get

$$\begin{aligned} W(\Gamma_{D_{2n}}) &= \left|V(\Gamma_{D_{2n}})\right| \left(\left|V(\Gamma_{D_{2n}})\right| - 1\right) - E(\Gamma_{D_{2n}}) \\ &= 2n(2n-1) - \frac{n^2+n}{2} \\ &= 4n^2 - 2n - \frac{n^2}{2} - \frac{n}{2} \\ &= \frac{8n^2 - 4n - n^2 - n}{2} \\ &= \frac{7n^2 - 5n}{2}. \end{aligned}$$

From previous research, the proof of Wiener index is as follows

$$\begin{aligned} \sum d(u,v) &= \sum_{v \in D_{2n} \setminus e} d(e,v) + \sum_{u,v \in \langle a \rangle} d(u,v) + \sum_{\substack{v \in \{b,\ldots,a^{n-1}b\} \\ u \in \langle a \rangle}} d(u,v) + \sum_{u,v \in \{b,\ldots,a^{n-1}b\}} d(u,v) \\ &= (2n-1)(1) + \binom{n-1}{2}(1) + (n-1)(n)(2) + \binom{n}{2}(2) \\ &= (2n-1) + \frac{(n-1)(n-2)}{2} + 2n^2 - 2n + n(n-1) \\ &= \frac{4n - 2 + n^2 - 3n + 2 + 4n^2 - 4n + 2n^2 - 2n}{2} \\ &= \frac{7n^2 - 5n}{2}. \end{aligned}$$

□

From the two proofs above, we get that the new approach is more easier because we can find the index with the total edges and vertices. Based on that, the new approach will be used in another indices.

Theorem 3.5. Hyper-Wiener Index
Let $\Gamma_{D_{2n}}$ is power graph of D_{2n}. If $n = p^k$, p is primes and $k \in \mathbb{Z}$. Then the Hyper-Wiener Index of $\Gamma_{D_{2n}}$ is

$$WW(\Gamma_{D_{2n}}) = 5n^2 - 4n.$$

Proof. Let $\Gamma_{D_{2n}}$ then $|V(\Gamma_{D_{2n}})| = 2n$. Based on Theorem 3.3 and Lemma 2.6 we get

$$\begin{aligned} WW(\Gamma_{D_{2n}}) &= \frac{3}{2}\left|V(\Gamma_{D_{2n}})\right|\left(\left|V(\Gamma_{D_{2n}})\right| - 1\right) - 2\left|E(\Gamma_{D_{2n}})\right| \\ &= \frac{3}{2}(2n)(2n-1) - 2\left(\frac{n^2+n}{2}\right) \\ &= 3n(2n-1) - n^2 - n \\ &= 6n^2 - 3n - n^2 - n \\ &= 5n^2 - 4n. \end{aligned}$$

□

Theorem 3.6. Harary Index
Let $\Gamma_{D_{2n}}$ is power graph of D_{2n}. If $n = p^k$, p is primes and $k \in \mathbb{Z}$. Then the Harary Index of $\Gamma_{D_{2n}}$ is

$$H(\Gamma_{D_{2n}}) = \frac{5n^2 - n}{4}.$$

Proof. Let $\Gamma_{D_{2n}}$ then $|V(\Gamma_{D_{2n}})| = 2n$. Based on Theorem 3.3 and Lemma 2.7 we get

$$\begin{aligned} H(\Gamma_{D_{2n}}) &= \frac{1}{4}\left|V(\Gamma_{D_{2n}})\right|\left(\left|V(\Gamma_{D_{2n}})\right| - 1\right) + \frac{1}{2}\left|E(\Gamma_{D_{2n}})\right| \\ &= \frac{1}{4}(2n)(2n-1) + \frac{1}{2}\left(\frac{n^2+n}{2}\right) \\ &= \frac{4n^2 - 2n + n^2 + n}{4} \\ &= \frac{5n^2 - n}{4}. \end{aligned}$$

□

To derive the theorem below, the total number of edges, the first Zagreb index, and the second Zagreb index are required.

Theorem 3.7. Gutman Index
Let $\Gamma_{D_{2n}}$ is power graph of D_{2n}. If $n = p^k$, p is primes and $k \in \mathbb{Z}$. Then Gutman Index of $\Gamma_{D_{2n}}$ is

$$Gut(\Gamma_{D_{2n}}) = \frac{(n^4 + n) + 3(n^3 - n^2)}{2}.$$

Proof. To proof this theorem, we use Theorem 3.3, Theorem 2.10, and Theorem 2.11. Then, we get

$$\begin{aligned}
Gut(\Gamma_{D_{2n}}) &= 4\left|E(\Gamma_{D_{2n}})\right| - M_1(\Gamma_{D_{2n}}) - M_2(\Gamma_{D_{2n}}) \\
&= 4\left(\frac{n^2+n}{2}\right)^2 - n^2(n+1) - \frac{n^4-n^3+3n^2-n}{2} \\
&= n^4+2n^3+n^2-n^3-n^2-\frac{n^4-n^3+3n^2-n}{2} \\
&= \frac{2n^4+2n^3-n^4+n^3-3n^2+n}{2} \\
&= \frac{n^4+3n^3-3n^2+n}{2} \\
&= \frac{(n^4+n)+3(n^3-n^2)}{2}.
\end{aligned}$$

□

4. Conclusion

This study introduces a simplified method for computing topological indices of the power graph of the dihedral group D_{2n} with $n = p^k$, p is primes and $k \in \mathbb{Z}$. Using the fact that the graph has diameter 2, the new approach allows these indices to be calculated directly from the numbers of vertices and edges. The results match those obtained using conventional methods, confirming that this new approach is both valid and more efficient.

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