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The Extended Metric Space on Max-Plus Algebra

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ABSTRACT

Max-Plus Algebra is the newly emerged mathematical object as one of the algebraic structures. Max Plus Algebra is a semi-ring with the maximum operation as its addition and the plus operation as its multiplication. In 2012, Carl et.al. established a novel notion about metric in max-plus geometry which is the semi-module over the semi-ring with maximum and addition biner operations. The writer researched to discover the distance function or metric, especially for the extended real-valued metric of Max-Plus Algebra and its properties with maximum and addition biner operations. By using both direct and indirect proof methods, the distance function of Max Plus Algebra and its topological properties were obtained.

Keywords: Max Plus Algebra, Metric Space, Sequence, Convergence

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1. Introduction

Max-Plus Algebra is a field of mathematics that introduces a distinct and innovative algebraic structure, differing from the conventional frameworks of classical algebra. It utilizes the operations of taking the maximum and addition, rather than traditional addition and multiplication, making it particularly useful for solving problems like optimization, operations research, and dynamic systems. The max-plus algebra set is $\mathbb{R}_\varepsilon =: \mathbb{R} \cup \{\varepsilon\}$ where $\varepsilon = -\infty$. Max-plus algebra is a semi-ring with the max operation of $\eta \oplus \zeta = \max(\eta, \zeta)$ and the plus operation of $\eta \otimes \zeta = \eta + \zeta$ for all $\eta, \zeta \in \mathbb{R}_\varepsilon$. Below are the properties of max-plus algebra as a semi-ring under max and operation. That is $\forall \eta, \zeta, z \in \mathbb{R}_\varepsilon$ holds:

1. $\eta \oplus \zeta \in \mathbb{R}_\varepsilon$
2. $\eta \oplus \zeta = \eta \oplus \zeta$
3. $\exists \varepsilon = -\infty \in \mathbb{R}_\varepsilon \ni \eta \oplus \varepsilon = \varepsilon \oplus \eta = \eta$
4. $\eta \oplus (\zeta \oplus \theta) = (\eta \oplus \zeta) \oplus \theta$
5. $\eta \otimes \zeta \in \mathbb{R}_\varepsilon$
6. $\eta \otimes \zeta = \zeta \otimes \eta$

7. $\eta \otimes \varepsilon = \varepsilon$
8. $\exists 0 \in \mathbb{R}_{\varepsilon} \ni \eta \otimes 0 = 0 \otimes \eta = \eta$
9. $\eta \otimes (\zeta \otimes \theta) = (\eta \otimes \zeta) \otimes \theta$
10. $\eta \otimes (\zeta \oplus \theta) = (\eta \otimes \zeta) \oplus (\eta \otimes \theta)$
11. $(\zeta \oplus \theta) \otimes \eta = (\zeta \otimes \eta) \oplus (\theta \otimes \eta)$.

Max-plus also has an element order such that, for every $\eta, \zeta \in \mathbb{R}_{\varepsilon}$, $\eta \succeq \zeta \iff \eta \oplus \zeta = \eta$. It's easy to see that $\eta \succeq \zeta \iff \eta \geq \zeta$ [1].

Max-plus algebra is widely applied in computing, optimization, simulation, mathematical modelling, and other fields. It is often said to be the linear algebra of discrete mathematics [2, 3]. Max plus is also used in creating discrete mathematical system models and their control analysis [4].

Sometimes, errors in models or optimizations using max-plus algebra are still difficult to do. Unlike real numbers where we can see the error using absolute values, the max-plus algebra does not have its absolute value. Absolute value is a metric. So it is necessary to define a metric in max-plus algebra.

Distance is a quantity that we usually encounter in our daily lives. Distance between cars, distance between houses, time distance, distance between two points, distance between two objects, and so on. The most fundamental object of mathematics is sets. The set itself consists of an empty set and a non-empty set. If we have a non-empty set with more than 1 member, then we can define the distance between two members of the set. In mathematics, the distance is called a metric [5].

Metrics are widely used to analyze the existence of fixed points [6] on a mapping such as contraction mapping, and so on [7, 8]. In addition, it is also used to define convergent sequences, limits, derivatives, and even integrals. If we have a function in the max-plus algebra, we need a metric to define the limit of a function, the derivative of a function, and even the integral of a function. We need to use a metric to know whether the function is continuous. In 2014, Carl et.al.[9] established a novel notion about metric in max-plus geometry which is the semi-module over semi-ring with maximum and addition biner operations. Therefore, the author researched to find the distance or metric function of the max-plus algebra and its properties. Since max-plus algebra has a negative infinity as an element of the set, instead of using the ordinary metric, we will to use the extended real-valued metric in this article.

2. Metric

The metric or distance that is popular is the absolute value of $\eta - \zeta$ whenever $\eta, \zeta \in \mathbb{R}$. The properties of absolute value are $\forall \eta, \zeta, \theta \in \mathbb{R}$ then :

1. $|\eta - \zeta| \geq 0$ (non-negativity)
2. $|\eta - \zeta| = 0 \iff \eta = \zeta$
3. $|\eta - \zeta| = |\zeta - \eta|$ (symmetry)
4. $|\eta - \zeta| \leq |\eta - \theta| + |\theta - \zeta|$ (triangle inequality).

Those properties would be generated to be the definition of metric for any non-empty set.

Definition 2.1. [10] Given a non-empty set X, a map $d : X \times X \to \mathbb{R}$ is said to be a metric of X if and only if $\forall \eta, \zeta, \theta \in X$ holds:

1. $d(\eta, \zeta) \geq 0$
2. $d(\eta, \zeta) = 0 \iff \eta = \zeta$
3. $d(\eta, \zeta) = d(\eta, \zeta)$
4. $d(\eta, \zeta) \leq d(\eta, \theta) + d(\theta, \zeta)$.

Then, (X, d) is called metric space. But, unlike the ordinary metric, there is an extended real-valued metric which uses the extended real number set ($\bar{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$) as the map of the metric.

Definition 2.2. A map $d : X \times X \to \bar{\mathbb{R}}$ is called the extended real-valued metric of X if d statisfied the metric properties.

From Definition 2.2 we can construct the extended real-valued metric of max-plus algebra and from now we call it extended metric of max-plus algebra.

Theorem 2.3. A map $d : \mathbb{R}_\varepsilon \times \mathbb{R}_\varepsilon \to \bar{\mathbb{R}}$ with

$$d(\eta, \zeta) = \begin{cases} (\eta \otimes -\zeta) \oplus (-\eta \otimes \zeta) & \text{, if } \eta, \zeta \in \mathbb{R}_\varepsilon \backslash \{\varepsilon\} \\ 0 & \text{, if } \eta = \zeta = \varepsilon \\ \infty & \text{, if } \eta = \varepsilon \text{ or } \zeta = \varepsilon \end{cases}, \tag{1}$$

where $\bar{\mathbb{R}} = \mathbb{R}_\varepsilon \cup \{\infty\}$ defined the extended metric of max-plus algebra.

Proof. It's obviously true for $\eta = \varepsilon$ or $\zeta = \varepsilon$. Therefore, take any $\eta, \zeta, \theta \in \mathbb{R}_\varepsilon \backslash \{\varepsilon\}$. First, we will show that $d(\eta, \zeta)$ is non-negative.

Case 1.

If $\eta \succeq \zeta$ then $\eta \otimes -\zeta \succeq 0 \iff \eta \otimes -\zeta \succeq 0$ and $0 \succeq -\eta \otimes \zeta \iff -\eta \otimes \zeta \preceq 0$. So that $(\eta \otimes -\zeta) \geq (-\eta \otimes \zeta)$, implies

$$\begin{aligned} d(\eta, \zeta) &= (\eta \otimes -\zeta) \oplus (-\eta \otimes \zeta) \\ &= \eta \otimes -\zeta \\ &\geq 0. \end{aligned}$$

Case 2.

If $\zeta \succeq \eta$ then $0 \succeq \eta \otimes -\zeta \iff \eta \otimes -\zeta \preceq 0$ and $-\eta \otimes \zeta \succeq 0 \iff -\eta \otimes \zeta \geq 0$. So that, $(\eta \otimes -\zeta) \leq (-\eta \otimes \zeta)$, implies

$$\begin{aligned} d(\eta, \zeta) &= (\eta \otimes -\zeta) \oplus (-\eta \otimes \zeta) \\ &= -\eta \otimes \zeta \\ &\geq 0. \end{aligned}$$

Hence, $d(\eta, \zeta), \forall \eta, \zeta \in \mathbb{R}_\varepsilon$ is non-negative.

Second, we want to prove that $d(\eta, \zeta) = 0 \iff \eta = \zeta$. $d(\eta, \zeta) = 0 \iff (\eta \otimes -\zeta) \oplus (-\eta \otimes \zeta) = 0 \iff \eta \otimes -\zeta = 0 \iff 0 \succeq -\eta \otimes \zeta$ or $-\eta \otimes \zeta = 0 \otimes 0 \succeq \eta \otimes -\zeta$. If $0 \succeq \eta \otimes -\zeta$, so we have $\eta \otimes -\zeta = 0 \iff \eta + (-\zeta) = 0 \iff \eta = \zeta$. In the same way, we have $0 \succeq \eta \otimes -\zeta \iff \eta = \zeta$.

Third, we want to prove that d is symmetry.

$$\begin{aligned} d(\eta, \zeta) &= (\eta \otimes -\zeta) \oplus (-\eta \otimes \zeta) \\ &= (-\eta \otimes \zeta) \oplus (\eta \otimes -\zeta) \\ &= (\zeta \otimes -\eta) \oplus (-\zeta \otimes \eta) \\ &= d(\eta, \zeta). \end{aligned}$$

Hence, d is symmetry.

Last, we want to prove that $d(\eta,\zeta) \leq d(\eta,\theta) + d(\theta,\zeta)$.

$$\begin{aligned} d(\eta,\theta)+d(\theta,\zeta) =& ((\eta\otimes-\theta)\oplus(-\eta\otimes\theta))\otimes((\theta\otimes-\zeta)\oplus(-\theta\otimes\zeta)) \\ =& ((\eta\otimes-\theta)\otimes(\theta\otimes-\zeta))\oplus((\eta\otimes-\theta)\otimes(-\theta\otimes\zeta))\oplus((-\eta\otimes\theta)\otimes(\theta\otimes-\zeta)) \\ &\oplus((-\eta\otimes\theta)\otimes(-\theta\otimes\zeta)) \\ =& ((\eta\otimes-\zeta)\oplus(-\eta\otimes\zeta))\oplus((\eta\otimes\zeta\otimes-\theta\otimes-\theta)\oplus(-\eta\otimes-\zeta\otimes\theta\otimes\theta)) \\ \geq& (\eta\otimes-\zeta)\oplus(-\eta\otimes\zeta) \\ =& d(\eta,\zeta). \end{aligned}$$

□

Now, we define a neighbourhood of $c \in \mathbb{R}_\varepsilon$. The neighbourhood of a point c are points whose distance to c is less than something that limits it.

Definition 2.4. Let $\delta > 0$ is a real number and d is an extended metric of $\mathbb{R}_\varepsilon$. A neighbourhood of the point $c \in \mathbb{R}_\varepsilon$ is define by

$$P_\delta(c) = \{\eta \in \mathbb{R}_\varepsilon | d(\eta,c) < \delta\}. \tag{2}$$

Let A be a nonempty set that contains c. If we take any neighbourhood of the point c on A, if it still contains some of the elements of A that are not c, we say c as a limit point in A.

Definition 2.5. A point c belongs to $\mathbb{R}_\varepsilon$ is a called limit point of $A \subseteq \mathbb{R}_\varepsilon$ if for all positive δ implies

$$P_\delta(c) \cap A \backslash \{c\} \neq \emptyset.$$

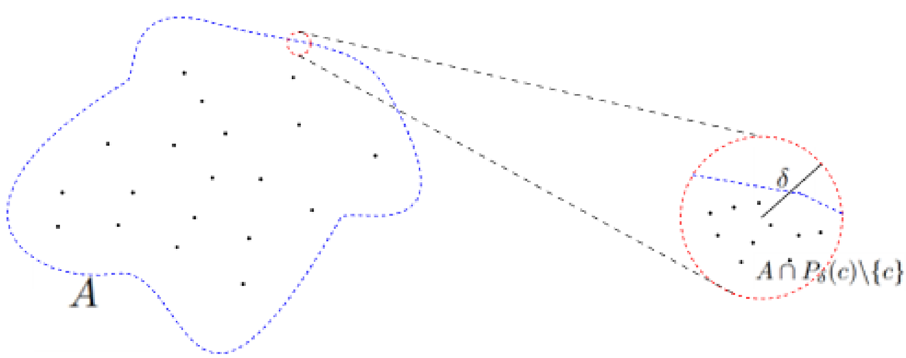

Figure 1. Limit point of A.

3. Sequence on Max-Plus Algebra

In real numbers, we know whether a sequence of real numbers is convergence or not. Here, we introduce the fundamental concept of convergence and divergence sequence in max-plus algebra.

Definition 3.1. [11] Let (η_n) be a sequence on $\mathbb{R}_\varepsilon$. (η_n) is said to be convergence to $\eta \in \mathbb{R}_\varepsilon$ if $\forall\delta > 0, \exists N \in \mathbb{N}$ such that $\forall n \geq N$ implies $d(\eta_n,\eta) < \delta$ and denoted by $\eta_n \to \eta$ or $\lim \eta_n = \eta$. A sequence that is not convergence is said divergence.

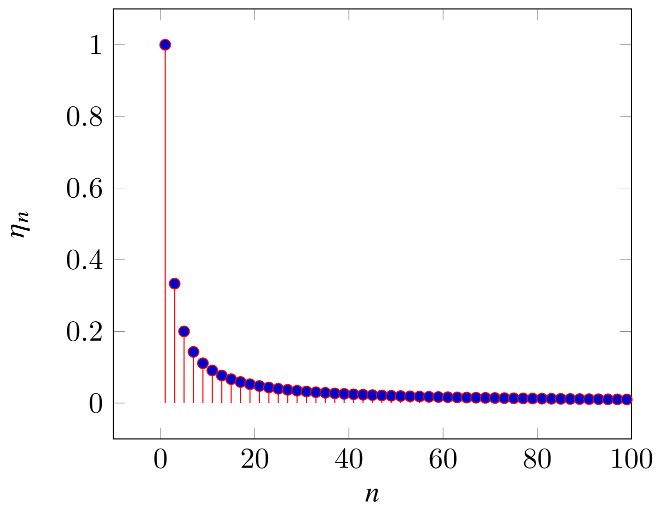

Figure 2. Graphic visualized of the sequence $\eta_n = \frac{100}{n}, \quad n \in \mathbb{N}$.

A convergent sequence is inherently connected to the idea of a limit point within its set. More precisely, a limit point of a set can be defined as a point that serves as the convergence point for at least one sequence within the set. In other words, the existence of a limit point implies that there is a sequence in the set that converges to that specific point. This relationship demonstrates the strong link between the convergence of sequences and the concept of limit points in the context of the set.

Corollary 3.2. *Let $c \in H \subseteq \mathbb{R}_\varepsilon$ is considered a limit point of a set H if and only if there exists a sequence (η_n) on H such that $\eta_n \to c$.*

The Corollary 3.2 means that if we have a sequence that convergence, the point that it converges is a limit point of A. For example, given a set $A = [\varepsilon, 0)$ with $(\eta_n) = \left\{-n^{\frac{9}{10}} \middle| n \in \mathbb{N}\right\}$. It is clear that $\eta_n \to \varepsilon$. Easy to check that ε is also a limit point of A.

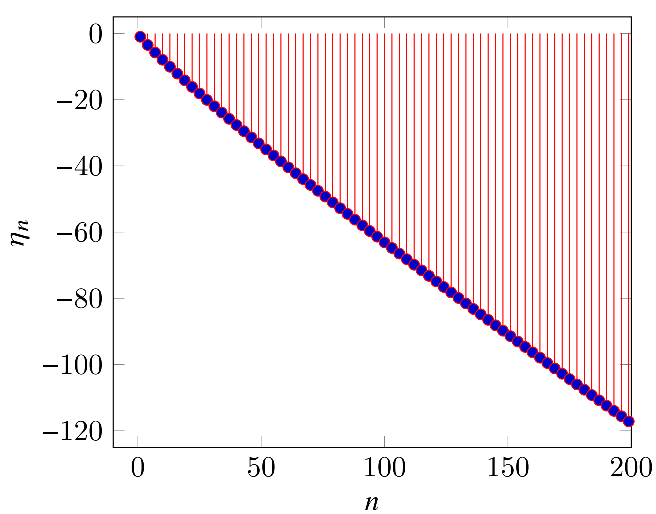

Figure 3. Graphic visualized of the sequence $\eta_n = -n^{\frac{9}{10}}$

References

[1] C. Wang, Y. Xia, and Y. Tao, "Ordered structures of polynomials over max-plus algebra," *Symmetry*, vol. 13, no. 7, p. 1137, 2021. https://doi.org/10.3390/sym13071137.

[2] P. Butkovič, "Max-algebra: the linear algebra of combinatorics?," *Linear Algebra and its applications*, vol. 367, pp. 313–335, 2003. https://doi.org/10.1016/S0024-3795(02)00655-9.

[3] J. Komenda, S. Lahaye, J.-L. Boimond, and T. van den Boom, "Max-plus algebra in the history of discrete event systems," *Annual Reviews in Control*, vol. 45, pp. 240–249, 2018. https://doi.org/10.1016/j.arcontrol.2018.04.004.

[4] B. De Schutter, T. van den Boom, J. Xu, and S. S. Farahani, "Analysis and control of max-plus linear discrete-event systems: An introduction," *Discrete Event Dynamic Systems*, vol. 30, pp. 25–54, 2020. https://doi.org/10.1007/s10626-019-00294-w.

[5] M. Masriani, Q. Aini, and S. Bahri, "Fuzzy metric space and its topological properties," *EIGEN MATHEMATICS JOURNAL*, pp. 74–79, 2021. https://doi.org/10.29303/emj.v4i2.95.

[6] L. Zicky, I. G. N. R. Usadha, *et al.*, "Fixed point theorem on fuzzy metric space," in *Journal of Physics: Conference Series*, vol. 1218, p. 012062, IOP Publishing, 2019. https://doi.org/10.1088/1742-6596/1218/1/012062.

[7] A. Ansar and H. Hikmah, "Eksistensi titik tetap pemetaan c-kontraktif diperumum pada ruang metrik parsial kompleks," *JMPM: Jurnal Matematika dan Pendidikan Matematika*, vol. 6, no. 2, pp. 90–101, 2021. https://doi.org/10.26594/jmpm.v6i2.2138.

[8] S. Sunarsini, S. Sadjidon, and S. E. Setiawan, "Teorema titik tetap untuk dua pemetaan di ruang metrik cone rectangular," *UNEJ e-Proceeding*, pp. 309–318, 2022.

[9] U. Carl, K. W. O'Neill, and N. Ryder, "Establishing a metric in max-plus geometry," *Rose-Hulman Undergraduate Mathematics Journal*, vol. 13, no. 2, p. 10, 2012. https://scholar.rose-hulman.edu/rhumj/vol13/iss2/10.

[10] A. Wahono and A. Ekayanti, "Pemetaan dalam ruang metrik-g," *Euclid*, vol. 6, no. 1, pp. 12–24, 2019. https://doi.org/10.33603/e.v6i1.1457.

[11] I. Kaplansky, *Set theory and metric spaces*, vol. 298. American Mathematical Society, 2020.