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Properties of Amalgamated Algebras Along an Ideal and Their Applications

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ABSTRACT

If given rings A and B, a ring homomorphism $f : A \to B$, and an ideal J of B, then a new ring can be constructed called amalgamated algebras along an ideal which is denoted by $A \bowtie^f J := \{(a, f(a) + j) \mid a \in A, j \in J\}$ with component-wise addition and multiplication. This paper discusses the construction, definition, properties such as isomporhisms, and characterization of amalgamated algebras along an ideal that is a prime ring and a Noetherian ring with detailed explanations. We also discuss its characterization as a reduced ring, which is a continuation from the previous paper. Furthermore, we investigate its characterization as an Artinian ring by adding an additional condition that every ideal of J has unity.

Keywords: amalgamated algebras, reduced ring, prime ring, Artinian ring.

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1. Introduction

One of research topics in algebra related to construction of a new ring from a given ring and an ideal is amalgamated algebras along an ideal. It was first introduced by Marco D'anna, et.al [1]. This construction is related to the construction by D.D Anderson [2]. Let A is a commutative ring with unity, X is a ring without unity and a module over A. Marco D'anna, et.al [1] define multiplicative operation as follows:

$$(a, x)(a', x') := (aa', ax' + a'x + xx')$$

for all $a, a' \in A$ and $x, x' \in X$. This construction also motivated by Dorroh [3] about embedding from ring without unity to ring with unity. Embedding from one ring to another ring can be shown by there exists monomorphism from ring to other ring. In their research, Marco D'anna, et.al [1] wrote about definition and construction of amalgamated algebras along an ideal. It can be viewed as a *"pullback"* or *"fiber product"* but the proof was skipped. Furthermore, they discussed about its properties including ideals and isomorphisms. Besides that, they also discussed its characterization as a reduced ring and a

Noetherian ring. Then, we make its characterization as a reduced ring with different statement which is a continuation from the previous paper. In addition, there is previous research by Nowakowska [4] which discussed characterization of amalgamated algebras along an ideal is a prime ring, thus we try to make more detailed explanations. Several proofs can be more detailed that motivate the author to carry out related research. We also investigate characterization of amalgamated algebras along an ideal is an Artinian ring which is not discussed in previous papers. We curious whether the properties not only applies in Noetherian rings but also in Artinian rings. We must give additional condition that every ideal of J has unity. Therefore, this research discusses about the construction, definition, properties, and characterization of amalgamated algebras along an ideal is a prime ring, a reduced ring, a Noetherian ring, and an Artinian ring.

We begin with definition of reduced ring.

Definition 1.1. ([5]) A ring R is called *reduced* if $r \in R$ and $r^2 = 0$ then $r = 0$. In other words, R only contains zero nilpotent element.

We want to prove a proposition from M. Atiyah [6] in which we are going to use to prove our results. In the previous paper, the proof of this proposition was skipped.

Proposition 1.2. ([6] Proposition 6.3 (i)) Let $0 \to M' \xrightarrow{\alpha} M \xrightarrow{\beta} M'' \to 0$ be an exact sequence of modules over A (commutative ring with unity). Then M is a Noetherian module if and only if M' and M'' are Noetherian modules.

Proof. ($\Rightarrow$) Let $L'_1 \subseteq L'_2 \subseteq \cdots$ be an ascending chain in M'. If $L'_i \subseteq L'_{i+1} \subset M'$, then $\alpha(L'_i) \subseteq \alpha(L'_{i+1}) \subset M$. As a result, an ascending chain is formed in M as follows: $\alpha(L'_1) \subseteq \alpha(L'_2) \subseteq \cdots$. Because M is a Noetherian module, there is a positive integer t, that satisfies

$$\alpha(L'_1) \subseteq \alpha(L'_2) \subseteq \cdots \subseteq \alpha(L'_t) = \alpha(L'_{t+1}) = \cdots .$$

Since α is injective, we obtain $L'_t = L'_{t+1}$. So, the chain satisfies ascending chain condition (eventually constant). Next, let $L''_1 \subseteq L''_2 \subseteq \cdots$ be an ascending chain in M''. If $L''_i \subseteq L''_{i+1} \subset M''$, then $\beta^{-1}(L''_i) \subseteq \beta^{-1}(L''_{i+1}) \subset M$. As a result, an ascending chain is formed in M as follows: $\beta^{-1}(L''_1) \subseteq \beta^{-1}(L''_2) \subseteq \cdots$. Because M is a Noetherian module, there is a positive integer k, that satisfies

$$\beta^{-1}(L''_1) \subseteq \beta^{-1}(L''_2) \subseteq \cdots \subseteq \beta^{-1}(L''_k) = \beta^{-1}(L''_{k+1}) = \cdots .$$

We only need to show $L''_{k+1} \subseteq L''_k$. Let any $y \in L''_{k+1}$. Since β is surjective, there exists $x \in M$ such that $\beta(x) = y$. We have $x \in \beta^{-1}(L''_{k+1})$. Furthermore, $x \in \beta^{-1}(L''_k)$. Thus, we obtain $\beta(x) = y \in L''_k$. Then $L''_k = L''_{k+1}$. It means that the chain satisfies ascending chain conditions (eventually constant).
($\Leftarrow$) Let $L_1 \subseteq L_2 \subseteq \cdots$ be an ascending chain in M. If $L_i \subseteq L_{i+1} \subset M$, then $\alpha^{-1}(L_i) \subseteq \alpha^{-1}(L_{i+1}) \subset M'$. As a result, an ascending chain is formed in M' as follows: $\alpha^{-1}(L_1) \subseteq \alpha^{-1}(L_2) \subseteq \cdots$. Because M' is a Noetherian module, there is a positive integer r, that satisfies

$$\alpha^{-1}(L_1) \subseteq \alpha^{-1}(L_2) \subseteq \cdots \subseteq \alpha^{-1}(L_r) = \alpha^{-1}(L_{r+1}) = \cdots .$$

Since α is injective, α^{-1} is also injective. We obtain $L_r = L_{r+1}$. It means that the chain satisfies ascending chain condition (eventually constant). □

Proposition 1.3. ([6] Proposition 6.3 (ii)) Let $0 \to M' \xrightarrow{\alpha} M \xrightarrow{\beta} M'' \to 0$ be an exact sequence of modules over A (commutative ring with unity). Then M is an Artinian module if and only if M' and M'' are Artinian modules.

Proof. ($\Rightarrow$) Let $L'_1 \supseteq L'_2 \supseteq \cdots$ be a descending chain in M'. If $L'_i \supseteq L'_{i+1}$, then $\alpha(L'_i) \supseteq \alpha(L'_{i+1})$. As a result, a descending chain is formed in M as follows: $\alpha(L'_1) \supseteq \alpha(L'_2) \supseteq \cdots$. Because M is an Artinian module, there is a positive integer k, that satisfies

$$\alpha(L'_1) \supseteq \alpha(L'_2) \supseteq \cdots \supseteq \alpha(L'_k) = \alpha(L'_{k+1}) = \cdots .$$

Since α is injective, we obtain $L'_k = L'_{k+1}$. So, the chain satisfies descending chain condition (eventually constant). Next, let $L''_1 \supseteq L''_2 \supseteq \cdots$ be a descending chain in M''. If $L''_i \supseteq L''_{i+1}$, then $\beta^{-1}(L''_i) \supseteq \beta^{-1}(L''_{i+1})$. As a result, a descending chain is formed in M as follows: $\beta^{-1}(L''_1) \supseteq \beta^{-1}(L''_2) \supseteq \cdots$. Because M is an Artinian module, there is a positive integer t, that satisfies

$$\beta^{-1}(L''_1) \supseteq \beta^{-1}(L''_2) \supseteq \cdots \supseteq \beta^{-1}(L''_t) = \beta^{-1}(L''_{t+1}) = \cdots .$$

We only need to show $L''_{t+1} \supseteq L''_t$. Let any $y \in L''_t$. Since β is surjective, there is $x \in M$ such that $\beta(x) = y$. We have $x \in \beta^{-1}(L''_t)$. Furthermore, $x \in \beta^{-1}(L''_{t+1})$. Thus, we obtain $\beta(x) = y \in L''_{t+1}$. Then $L''_t = L''_{t+1}$. It means that the chain satisfies descending chain condition (eventually constant).
($\Leftarrow$) Let $L_1 \supseteq L_2 \supseteq \cdots$ be a descending chain in M. If $L_i \supseteq L_{i+1}$, then $\alpha^{-1}(L_i) \supseteq \alpha^{-1}(L_{i+1})$. As a result, a descending chain is formed in M' as follows: $\alpha^{-1}(L_1) \supseteq \alpha^{-1}(L_2) \supseteq \cdots$. Because M' is an Artinian module, there is a positive integer s, that satisfies

$$\alpha^{-1}(L_1) \supseteq \alpha^{-1}(L_2) \supseteq \cdots \supseteq \alpha^{-1}(L_s) = \alpha^{-1}(L_{s+1}) = \cdots .$$

Since α is injective, α^{-1} is also injective. We obtain $L_s = L_{s+1}$. It means that the chain satisfies descending chain condition (eventually constant). □

Proposition 1.4. Let R is a ring. If I is an ideal of J, J is an ideal of R, and I has unity, then I is an ideal of R.

Proof. Let $r \in R$ and $i \in I$. Suppose that I has unity e. Since $e \in I \subseteq J$ and J is ideal of R, we have $re, er \in J$.

Note that, since I is ideal of J we obtain:

$$ri = r(ei) = (re)i \in I \quad \text{(Left Ideal)}$$

and

$$ir = (ie)r = i(er) \in I \quad \text{(Right Ideal)}.$$

Therefore, I is an ideal of R. □

2. Construction and Definition

Suppose that A is a commutative ring with unity, R is a ring without unity and a module over A. Since A and R are modules over A, we can form a set:

$$A \times R = \{(a, r) \mid a \in A, r \in R\}.$$

On this set, the following operations are defined:

$$(a_1, r_1) + (a_2, r_2) = (a_1 + a_2, r_1 + r_2)$$

and

$$a \cdot (a_1, r_1) = (aa_1, ar_1)$$

for all $(a_1, r_1), (a_2, r_2) \in A \times R$ and $a \in A$. We will show that $A \times R$ is a module over A.

1. First, we will show that $(A \times R, +)$ is an abelian group.
   (a) Let $(a_1, r_1), (a_2, r_2), (a_3, r_3) \in A \times R$. Then

$$\begin{aligned}(a_1, r_1) + ((a_2, r_2) + (a_3, r_3)) &= (a_1, r_1) + (a_2 + a_3, r_2 + r_3)\\ &= (a_1 + a_2 + a_3, r_1 + r_2 + r_3)\\ &= (a_1 + a_2, r_1 + r_2) + (a_3, r_3)\\ &= ((a_1, r_1) + (a_2, r_2)) + (a_3, r_3).\end{aligned}$$

   Therefore, the additive operation is associative.
   (b) There is $(0, 0) \in A \times R$ such that for any $(a, r) \in A \times R$ satisfies:

$$(a, r) + (0, 0) = (a + 0, r + 0) = (a, r)$$

   and

$$(0, 0) + (a, r) = (0 + a, 0 + r) = (a, r).$$

   (c) Let $(a, r) \in A \times R$, then there is $(a, r)^{-1} = (-a, -r)$ such that:

$$(a, r) + (-a, -r) = (a + (-a), r + (-r)) = (0, 0)$$

   and

$$(-a, -r) + (a, r) = (-a + a, -r + r) = (0, 0).$$

   (d) Let $(a_1, r_1), (a_2, r_2) \in A \times R$. Then

$$\begin{aligned}(a_1, r_1) + (a_2, r_2) &= (a_1 + a_2, r_1 + r_2)\\ &= (a_2 + a_1, r_2 + r_1)\\ &= (a_2, r_2) + (a_1, r_1).\end{aligned}$$

   Therefore, the additive operation is commutative.
2. We will show the abelian group $A \times R$ and the scalar multiplication operation $\cdot$ satisfy the axiom of module over A. We only need to show that $A \times R$ is a left module over A because A is a commutative ring. Let $(a_1, r_1), (a_2, r_2) \in A \times R$ and $a, a' \in A$. Then

$$\begin{aligned}a \cdot ((a_1, r_1) + (a_2, r_2)) &= a \cdot (a_1 + a_2, r_1 + r_2)\\ &= (aa_1 + aa_2, ar_1 + ar_2)\\ &= (aa_1, ar_1) + (aa_2, ar_2)\\ &= a \cdot (a_1, r_1) + a \cdot (a_2, r_2);\end{aligned}$$

$$\begin{aligned}(a + a') \cdot (a_1, r_1) &= ((a + a')a_1, (a + a')r_1)\\ &= (aa_1 + a'a_1, ar_1 + a'r_1)\\ &= (aa_1, ar_1) + (a'a_1, a'r_1)\\ &= a \cdot (a_1, r_1) + a' \cdot (a_1, r_1);\end{aligned}$$

$$\begin{aligned}(aa') \cdot (a_1, r_1) &= (aa'a_1, aa'r_1)\\ &= (a \cdot (a'a_1, a'r_1))\\ &= (a \cdot (a' \cdot (a_1, r_1));\end{aligned}$$

and

$$\begin{aligned} 1 \cdot (a_1, r_1) &= (1a_1, 1r_1) \\ &= (a_1, r_1). \end{aligned}$$

Thus, $A \times R$ is a module over A. Furthermore, the module is named a direct sum of A and R which is denoted by $A \oplus R$. The multiplicative operation below was defined in [1] :

$$(a_1, r_1)(a_2, r_2) := (a_1a_2, a_1r_2 + a_2r_1 + r_1r_2)$$

for all $a_1, a_2 \in A$ and $r_1, r_2 \in R$. Furthermore, in that paper, $A\dot{\oplus}R$ denotes direct sum $A \oplus R$ equipped by the multiplicative operation defined over.

We take several points from the lemma in the previous paper [1] that are related to our research purposes.

Lemma 2.1. ([1] Lemma 2.1)

1. $A\dot{\oplus}R$ is a ring with unity $(1, 0)$ and ring A can be embedded in the ring $A\dot{\oplus}R$ where $i_A : A \longrightarrow A\dot{\oplus}R$ defined by $a \mapsto (a, 0)$ for all $a \in A$.
2. R is an ideal in $A\dot{\oplus}R$.
3. If $p_A : A\dot{\oplus}R \mapsto A$ is a canonical projection defined by $(a, x) \mapsto a$ for all $a \in A$ and $x \in R$ then

$$0 \longrightarrow R \xrightarrow{i_R} A\dot{\oplus}R \xrightarrow{p_A} A \longrightarrow 0$$

is a split exact sequence of module over A.

Proof. We will prove this lemma which is skipped in the previous paper.

1. (a) $A\dot{\oplus}R$ is an abelian group since $A\dot{\oplus}R$ is a module over A.
   (b) Let $(a_1, r_1), (a_2, r_2), (a_3, r_3) \in A\dot{\oplus}R$. Then

$$(a_1, r_1)(a_2, r_2) = (a_1a_2, a_1r_2 + a_2r_1 + r_1r_2) \in A\dot{\oplus}R$$

and

$$\begin{aligned} ((a_1, r_1)(a_2, r_2))(a_3, r_3) &= (a_1a_2, a_1r_2 + a_2r_1 + r_1r_2)(a_3, r_3) \\ &= (a_1a_2a_3, a_1a_2r_3 + a_3a_1r_2 + a_3a_2r_1 + \\ &\quad a_3r_1r_2 + a_1r_2r_3 + a_2r_1r_3 + r_1r_2r_3) \\ &= (a_1a_2a_3, a_1a_2r_3 + a_1a_3r_2 + a_1r_2r_3 + \\ &\quad a_3r_2r_1 + r_1r_2r_3) \\ &= (a_1, r_1)(a_2a_3, a_2r_3 + a_3r_2 + r_2r_3) \\ &= (a_1, r_1)((a_2, r_2)(a_3, r_3)). \end{aligned}$$

Thus, $A\dot{\oplus}R$ is a semigroup under multiplication.

(c) Let $(a_1, r_1), (a_2, r_2), (a_3, r_3) \in A\dot{\oplus}R$. Then

$$\begin{aligned}(a_1, r_1)((a_2, r_2) + (a_3, r_3)) &= (a_1, r_1)(a_2 + a_3, r_2 + r_3)\\ &= (a_1a_2 + a_1a_3,\ a_1r_2 + a_1r_3 + a_2r_1 +\\ &a_3r_1 + r_1r_2 + r_1r_3)\\ &= (a_1a_2, a_1r_2 + a_2r_1 + r_1r_2) +\\ &(a_1a_3, a_1r_3 + a_3r_1 + r_1r_3)\\ &= (a_1, r_1)(a_2, r_2) + (a_1, r_1)(a_3, r_3)\end{aligned}$$

and

$$\begin{aligned}((a_1, r_1) + (a_2, r_2))(a_3, r_3) &= (a_1 + a_2, r_1 + r_2)(a_3, r_3)\\ &= (a_1a_3 + a_2a_3, a_1r_3 + a_2r_3 + a_3r_1 +\\ &a_3r_2 + r_1r_3 + r_2r_3)\\ &= (a_1a_3, a_1r_3 + a_3r_1 + r_1r_3) +\\ &(a_2a_3, a_2r_3 + a_3r_2 + r_2r_3)\\ &= (a_1, r_1)(a_3, r_3) + (a_2, r_2)(a_3, r_3).\end{aligned}$$

Thus, the distributive operation are satisfied.

Therefore, $A\dot{\oplus}R$ is a ring under additive and multiplicative operations.

Let $(a, r) \in A\dot{\oplus}R$. Then

$$(1, 0)(a, r) = (1a, 1r + a0 + 0r) = (a, r)$$

and

$$(a, r)(1, 0) = (a1, a0 + 1r + r0) = (a, r).$$

Thus, $A\dot{\oplus}R$ has unity $(1, 0)$.

Let $a_1, a_2 \in A$. Then

$$i_A(a_1 + a_2) = (a_1 + a_2, 0) = (a_1, 0) + (a_2, 0) = i_A(a_1) + i_A(a_2)$$

and

$$\begin{aligned}i_A(a_1a_2) &= (a_1a_2, 0)\\ &= (a_1a_2, a_10 + a_20 + 00)\\ &= (a_1, 0)(a_2, 0)\\ &= i_A(a_1)i_A(a_2)\end{aligned}$$

Thus, i_A is a ring homomorphism. Furthermore,

$$Ker(i_A) = \{a_1 \in A \mid i(a_1) = (0, 0)\} = \{a_1 \in A \mid (a_1, 0) = (0, 0)\} = \{0\}.$$

Therefore, A is embedded in $A\dot{\oplus}R$.

2. To show that R is an ideal in $A\dot{\oplus}R$, it is enough to show that there is a monomorphism from R to $A\dot{\oplus}R$. We form a mapping $i_R : R \to A\dot{\oplus}R$ defined by $i_R(r) = (0, r)$ for every $r \in R$. Let $r_1, r_2 \in R$. Then

$$\begin{aligned} i_R(r_1 + r_2) &= (0, r_1 + r_2) \\ &= (0, r_1) + (0, r_2) \\ &= i_R(r_1) + i_R(r_2) \end{aligned}$$

and

$$\begin{aligned} i_R(r_1 r_2) &= (0, r_1 r_2) \\ &= (00, 0r_2 + 0r_1 + r_1 r_2) \\ &= (0, r_1)(0, r_2) \\ &= i_R(r_1) i_R(r_2). \end{aligned}$$

Furthermore,

$$\begin{aligned} Ker(i_R) &= \{r_1 \in R \mid i_R(r_1) = (0,0)\} \\ &= \{r_1 \in R \mid (0, r_1) = (0,0)\} \\ &= \{0\}. \end{aligned}$$

3. Let $p_A : A\dot{\oplus}R \mapsto A$ be a canonical projection. We will prove that, the sequence $0 \to R \xrightarrow{i_R} A\dot{\oplus}R \xrightarrow{p_A} A \to 0$ is a split exact sequence of module over A.
   (a) We already know that R and $A\dot{\oplus}R$ are modules over A. We only need to show that A is a module over A. Since A is a ring with unity, it is clear that A is a module over itself.
   (b) i. Let $r_1, r_2 \in R$ and $a \in A$. Then

$$\begin{aligned} i_R(r_1 + r_2) &= (0, r_1 + r_2) \\ &= (0, r_1) + (0, r_2) \\ &= i_R(r_1) + i_R(r_2) \end{aligned}$$

and

$$\begin{aligned} i_R(ar_1) &= (0, ar_1) \\ &= a(0, r_1) \\ &= a i_R(r_1). \end{aligned}$$

Therefore, i_R is a module homomorphism over A.

   ii. Let $(a_1, r_1), (a_2, r_2) \in A\dot{\oplus}R$ and $a \in A$. Then

$$\begin{aligned} p_A((a_1, r_1) + (a_2, r_2)) &= p_A((a_1 + a_2, r_1 + r_2)) \\ &= a_1 + a_2 \\ &= p_A(a_1, r_1) + p_A(a_2, r_2) \end{aligned}$$

and

$$\begin{aligned} p_A(a(a_1, r_1)) &= p_A(aa_1, ar_1) \\ &= aa_1 \\ &= a p_A(a_1, r_1). \end{aligned}$$

Therefore, p_A is a module homomorphism over A.

(c) i. It is given from previous point (2) that i_R is a ring monomorphism.
ii. Let $(a, r) \in A\dot{\oplus}R$. Then

$$\begin{aligned} Im(i_R) &= \{(a, r) \in A\dot{\oplus}R \mid i(r_1) = (a, r), \quad \text{for some } r_1 \in R\} \\ &= \{(a, r) \in A\dot{\oplus}R \mid (0, r_1) = (a, r)\} \\ &= \{(a, r) \in A\dot{\oplus}R \mid a = 0, r = r_1\} \\ &= \{(0, r) \mid r \in R\} \end{aligned}$$

and

$$\begin{aligned} Ker(p_A) &= \{(a, r) \in A\dot{\oplus}R \mid p_A(a, r) = 0_A\} \\ &= \{(a, r) \in A\dot{\oplus}R \mid a = 0_A\} \\ &= \{(0, r) \mid r \in R\}. \end{aligned}$$

Thus, $Im(i_R) = Ker(p_A)$.
iii. Let any $y \in A$. We will prove that there exists $(a, r) \in A\dot{\oplus}R$ such that $p_A(a, r) = y$. Note that based on the definition $p_A(y, r) = y$ for any $r \in R$. Therefore, for every $y \in A$, there exists $(a, r) \in A\dot{\oplus}R$, namely $(a, r) = (y, r)$ such that, $p_A(a, r) = y$. Hence, p_A is surjective.

From $(a), (b), (c)$ we prove that the sequence is an exact sequence of module over A. Now, we will show that the exact sequence is split. We know that p_A is a module homomorphism. Morover, it has been proven previously at point (1) that i_A is a module homomorphism. Thus, we only need to show that $p_A \circ i_A = 1_A$, where 1_A is the identity homomorphism in A. Let $a \in A$. Then

$$(p_A \circ i_A)(a) = p_A(i_A(a)) = p_A((a, 0)) = a.$$

We get $p_A \circ i_A = 1_A$.

□

Lemma 2.2. ([1] Lemma 2.3) Let $f : A \to B$ be a ring homomorphism and J be an ideal of B.

1. $A\dot{\oplus}J$ is a ring.
2. The mapping $f^{\bowtie} : A\dot{\oplus}J \to A \times B$ defined by $(a, j) \mapsto (a, f(a) + j)$ for all $a \in A$ and $j \in J$ is a ring monomorphism.
3. The mapping $i_A : A \to A\dot{\oplus}J$ defined by $a \mapsto (a, 0)$ for all $a \in A$ and $i_J : J \to A\dot{\oplus}J$ defined by $j \mapsto (0, j)$ for all $j \in J$ are ring monomorphisms and module monomorphisms over A.
4. Let $p_A : A\dot{\oplus}J \to A$ is canonical projection defined by $(a, j) \mapsto a$ for all $a \in A$ and $j \in J$. Then

$$0 \to J \xrightarrow{i_J} A\dot{\oplus}J \xrightarrow{p_A} A \to 0$$

is a split exact sequence of module over A.

Proof. We will prove this lemma in which the proof is skipped in the previous paper. The ideal J is also J a module over A since f induces J as a natural structure of the module. Then $a \cdot j := f(a)j$ for all $a \in A$ and $j \in J$.

1. Based on Lemma 2.1 (1) and since J is a module over A, we have $A\dot{\oplus}J$ is also a ring.
2. Let $(a_1, j_1), (a_2, j_2) \in A\dot{\oplus}J$. Then

$$\begin{aligned} f^{\bowtie}((a_1,j_1)+(a_2,j_2)) &= f^{\bowtie}(a_1+a_2, j_1+j_2) \\ &= (a_1+a_2, (f(a_1+a_2)) + (j_1+j_2)) \\ &= (a_1+a_2, f(a_1)+f(a_2)+j_1+j_2) \\ &= (a_1+a_2, f(a_1)+j_1+f(a_2)+j_2) \\ &= (a_1, f(a_1)+j_1) + (a_2, f(a_2)+j_2) \\ &= f^{\bowtie}(a_1,j_1) + f^{\bowtie}(a_2,j_2) \end{aligned}$$

and

$$\begin{aligned} f^{\bowtie}((a_1,j_1)(a_2,j_2)) &= f^{\bowtie}(a_1a_2, a_1j_2+a_2j_1+j_1j_2) \\ &= (a_1a_2, (f(a_1a_2)) + (a_1j_2+a_2j_1+j_1j_2)) \\ &= (a_1a_2, f(a_1)f(a_2)+a_1j_2+a_2j_1+j_1j_2) \\ &= (a_1a_2, f(a_1)f(a_2)+f(a_1)j_2+j_1f(a_2)+j_1j_2) \\ &= (a_1, f(a_1)+j_1)(a_2, f(a_2)+j_2) \\ &= f^{\bowtie}(a_1,j_1)f^{\bowtie}(a_2,j_2). \end{aligned}$$

Moreover,

$$\begin{aligned} Ker(f^{\bowtie}) &= \{(a_1,j_1) \in A\dot{\oplus}J \mid f^{\bowtie}((a_1,j_1)) = (0,0)_{A\times B}\} \\ &= \{(a_1,j_1) \in A\dot{\oplus}J \mid (a_1, f(a_1)+j_1) = (0,0)_{A\times B}\} \\ &= \{(a_1,j_1) \in A\dot{\oplus}J \mid a_1 = 0,\ j_1 = -f(a_1) = 0\} \\ &= \{(0,0)\}. \end{aligned}$$

Therefore, $f^{\bowtie}$ is a ring monomorphism.

3. (a) Since J is also a module over A and based on Lemma 2.1 (1), it is clear that the mapping i_A is also a monomorphism.
   (b) Since J is also a module over A and based on Lemma 2.1 (2), it is clear that the mapping i_J is also a monomorphism.
4. Based on Lemma 2.1 (3) where R is a module over A, and since J is also a module over A, it is clear that the sequence $0 \longrightarrow J \xrightarrow{i_J} A\dot{\oplus}J \xrightarrow{p_A} A \longrightarrow 0$ is a split exact sequence module over A.

□

Since $A\dot{\oplus}J$ is a ring, D'anna Marco, et.al [1] define:

$$A\bowtie^f J := f^{\bowtie}(A\dot{\oplus}J) = \{(a, f(a)+j) \mid a \in A, j \in J\}.$$

It is called **amalgamation of A with B along J with respect to** $f : A \longrightarrow B$. To prove $A \bowtie^f J$ is a ring, we only prove that $A \bowtie^f J$ is subring of $A \times B$. Let $(a, f(a)+j), (a', f(a')+j') \in A \bowtie^f J$. Then

$$(a, f(a)+j) - (a', f(a')+j') = (a-a', f(a-a') + (j-j')) \in A \bowtie^f J$$

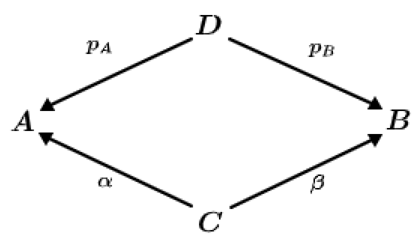
and

$$(a, f(a)+j)(a', f(a')+j') = (aa', f(aa') + f(a)j' + jf(a') + jj') \in A \bowtie^f J.$$

3. Pullback Construction

Definition 3.1. ([1]*Definition* 4.1) The set $D := \alpha \times_C \beta := \{(a,b) \in A \times B \mid \alpha(a) = \beta(b)\}$ is named *pullback or fiber product* of α and β, where $\alpha : A \to$ and $\beta : B \to C$ are ring homomorphisms.

The image below is pullback construction in diagram form.



The mapping $p_A : D \to A$ is defined by $(a,b) \mapsto a$ and $p_B : D \to B$ is defined by $(a,b) \mapsto b$. We will show that D is a subring of $A \times B$. Let $(a,b),(a',b') \in D$. Then $\alpha(a) = \beta(b)$ and $\alpha(a') = \beta(b')$. Thus,

$$(a,b) - (a',b') = (a - a', b - b') \in D$$

because $\alpha(a - a') = \alpha(a) - \alpha(a') = \beta(b) - \beta(b') = \beta(b - b')$. Also,

$$(a,b)(a',b') = (aa', bb') \in D$$

because $\alpha(aa') = \alpha(a)\alpha(a') = \beta(b)\beta(b') = \beta(bb')$.

Since D and $A \bowtie^f J$ are subring of $A \times B$, the amalgamated algebras along an ideal can be seen as pullback or fiber product which is explained in the proposition below.

Proposition 3.2. ([1]*Proposition* 4.2) Let $f : A \to B$ be a ring homomorphism and J be an ideal of B. Then, $A \bowtie^f J = \breve{f} \times_{B/J} \pi$ where $\pi : B \to B/J$ and $\breve{f} := \pi \circ f$.

Proof. Let $a \in A$ and $b \in B$. Based on the Definition 3.1, we get the following:

$$\begin{aligned}
\breve{f} \times_{B/J} \pi &= \{(a,b) \mid \breve{f}(a) = \pi(b)\} \\
&= \{(a,b) \mid \pi(f(a)) = b + J\} \\
&= \{(a,b) \mid f(a) + J = b + J\} \\
&= \{(a,b) \mid f(a) - b \in J\} \\
&= \{(a,b) \mid f(a) - b = j, \text{for some } j \in J\} \\
&= \{(a, f(a) + j) \mid a \in A, j \in J\} \\
&= A \bowtie^f J.
\end{aligned}$$

□

We can characterize D as a Noetherian ring which is explained below.

Proposition 3.3. ([1]*Proposition* 4.10) With notation from Definition 3.1, D is a Noetherian ring if only if $Ker(\beta)$ and $p_A(D)$ are Noetherian rings (Noetherian modules over D).

Proof. Based on the Proposition 1.2, we form

$$0 \longrightarrow Ker(\beta) \xrightarrow{i} D \xrightarrow{p_A} p_A(D) \longrightarrow 0$$

where $i : Ker(\beta) \longrightarrow D$ defined by $x \mapsto (0, x)$ for all $x \in Ker(\beta)$. It suffices to show that the sequence is an exact sequence of module over D.

1. (a) We will show that $Ker(\beta)$ is a module over D. First, we will prove that $Ker(\beta)$ is an abelian group. It is clear $Ker(\beta) \subseteq B$ where B is a ring and $Ker(\beta) \neq \emptyset$ since $\exists 0 \in Ker(\beta)$. Let $x_1, x_2 \in Ker(\beta)$. Then

$$\beta(x_1 - x_2) = \beta(x_1) - \beta(x_2) = 0 - 0 = 0.$$

We get $x_1 - x_2 \in Ker(\beta)$. Next, we define the scalar multiplication operation,

$$\cdot : D \times Ker(\beta) \longrightarrow Ker(\beta)$$

$$((a, b), x) \mapsto bx$$

and

$$\cdot : Ker(\beta) \times D \longrightarrow Ker(\beta)$$

$$(x, (a, b)) \mapsto xb$$

for all $(a, b) \in D$ and $x \in Ker(\beta)$. Note that, $bx \in Ker(\beta)$ since $\beta(bx) = \beta(b)\beta(x) = \beta(b)0 = 0$ and $xb \in Ker(\beta)$ since $\beta(xb) = \beta(x)\beta(b) = 0\beta(b) = 0$. We will show that $Ker(\beta)$ and the scalar multiplication operation $\cdot$ satisfy the axioms of left module over D.
Let $x_1, x_2 \in Ker(\beta)$ and $(a_1, b_1), (a_2, b_2) \in D$. Then

$$\begin{aligned}(a_1, b_1) \cdot (x_1 + x_2) &= b_1(x_1 + x_2) \\ &= b_1x_1 + b_1x_2 \\ &= (a_1, b_1) \cdot x_1 + (a_1, b_1) \cdot x_2;\end{aligned}$$

$$\begin{aligned}((a_1, b_1) + (a_2, b_2)) \cdot x_1 &= (a_1 + a_2, b_1 + b_2) \cdot x_1 \\ &= (b_1 + b_2)x_1 \\ &= b_1x_1 + b_2x_1 \\ &= (a_1, b_1) \cdot x_1 + (a_2, b_2) \cdot x_1;\end{aligned}$$

$$\begin{aligned}((a_1, b_1)(a_2, b_2)) \cdot x_1 &= (a_1a_2, b_1b_2) \cdot x_1 \\ &= b_1b_2x_1 \\ &= (a_1, b_1) \cdot (b_2x_1) \\ &= (a_1, b_1) \cdot ((a_2, b_2) \cdot x_1);\end{aligned}$$

and

$$\begin{aligned}(1, 1) \cdot x_1 &= 1x_1 \\ &= x_1.\end{aligned}$$

Next, we will prove that $Ker(\beta)$ and the scalar multiplication operation $\cdot$ satisfy the axioms of right module over D.
Let $x_1, x_2 \in Ker(\beta)$ and $(a_1, b_1), (a_2, b_2) \in D$. Then

$$\begin{aligned}(x_1 + x_2) \cdot (a_1, b_1) &= (x_1 + x_2)b_1 \\ &= x_1 b_1 + x_2 b_1 \\ &= x_1 \cdot (a_1, b_1) + x_2 \cdot (a_1, b_1);\end{aligned}$$

$$\begin{aligned}x_1 \cdot ((a_1, b_1) + (a_2, b_2)) &= x_1 \cdot (a_1 + a_2, b_1 + b_2) \\ &= x_1(b_1 + b_2) \\ &= x_1 b_1 + x_1 b_2 \\ &= x_1 \cdot (a_1, b_1) \ + x_1 \cdot (a_2, b_2);\end{aligned}$$

$$\begin{aligned}x_1 \cdot ((a_1, b_1)(a_2, b_2)) &= x_1 \cdot (a_1 a_2, b_1 b_2) \\ &= x_1 b_1 b_2 \\ &= x_1 b_1 \cdot (a_2, b_2) \\ &= (x_1 \cdot (a_1, b_1)) \cdot (a_2, b_2);\end{aligned}$$

and

$$\begin{aligned}x_1 \cdot (1, 1) &= x_1 1 \\ &= x_1.\end{aligned}$$

Thus, $Ker(\beta)$ is a module over D.

(b) Since D is a ring, clearly D is a module over itself.

(c) We will show that $p_A(D)$ is a module over D. First, we will prove that $p_A(D)$ is an abelian group. It is clear that $p_A(D) \subseteq A$ where A is a ring and $p_A(D) \neq \emptyset$ since $\exists 0 \in p_A(D)$. Let $y_1, y_2 \in p_A(D)$, it means that $\exists (y_1, x_1), (y_2, x_2) \in D$ for some $x_1, x_2 \in B$. Note that,

$$(y_1, x_1) - (y_2, x_2) = (y_1 - y_2, x_1 - x_2) \in D.$$

We get $y_1 - y_2 \in p_A(D)$. Thus, $p_A(D)$ is an abelian group. Next, we define the scalar multiplication operation,

$$\begin{aligned}\cdot : D \times p_A(D) &\to p_A(D) \\ ((a, b), y) &\mapsto ay\end{aligned}$$

and

$$\begin{aligned}\cdot : p_A(D) \times D &\to p_A(D) \\ (y, (a, b)) &\mapsto ya\end{aligned}$$

for all $(a, b), (y, x) \in D$ means $a, y \in p_A(D)$. Note that, $(a, b)(y, x) = (ay, bx) \in D$; thus, $ay \in p_A(D)$ and $(y, x)(a, b) = (ya, xb) \in D$. Hence, $ya \in p_A(D)$. We will show that $p_A(D)$ and the scalar multiplication operation $\cdot$ satisfy the axioms of left module over D.
Let $y_1, y_2 \in p_A(D)$ and $(a_1, b_1), (a_2, b_2) \in D$. Then

$$\begin{aligned}(a_1, b_1) \cdot (y_1 + y_2) &= a_1(y_1 + y_2) \\ &= a_1 y_1 + a_1 y_2 \\ &= (a_1, b_1) \cdot y_1 + (a_1, b_1) \cdot y_2;\end{aligned}$$

$$\begin{aligned}((a_1,b_1)+(a_2,b_2))\cdot y_1 &= (a_1+a_2,b_1+b_2)\cdot y_1\\ &= (a_1+a_2)y_1\\ &= a_1y_1+a_2y_1\\ &= (a_1,b_1)\cdot y_1+(a_2,b_2)\cdot y_1;\end{aligned}$$

$$\begin{aligned}((a_1,b_1)(a_2,b_2))\cdot y_1 &= (a_1a_2,b_1b_2)\cdot y_1\\ &= a_1a_2y_1\\ &= (a_1,b_1)\cdot(a_2y_1)\\ &= (a_1,b_1)\cdot((a_2,b_2)\cdot y_1);\end{aligned}$$

and

$$\begin{aligned}(1,1)\cdot y_1 &= 1y_1\\ &= y_1.\end{aligned}$$

Next, we will prove that $p_A(D)$ and the scalar multiplication operation $\cdot$ satisfy the axioms of right module over D.
Let $y_1,y_2\in p_A(D)$ and $(a_1,b_1),(a_2,b_2)\in D$. Then

$$\begin{aligned}(y_1+y_2)\cdot(a_1,b_1) &= (y_1+y_2)a_1\\ &= y_1a_1+y_2a_1\\ &= y_1\cdot(a_1,b_1)+y_2\cdot(a_1,b_1);\end{aligned}$$

$$\begin{aligned}y_1\cdot((a_1,b_1)+(a_2,b_2)) &= y_1\cdot(a_1+a_2,b_1+b_2)\\ &= y_1(a_1+a_2)\\ &= y_1a_1+y_1a_2\\ &= y_1\cdot(a_1,b_1)\ +y_1\cdot(a_2,b_2);\end{aligned}$$

$$\begin{aligned}y_1\cdot((a_1,b_1)(a_2,b_2)) &= y_1\cdot(a_1a_2,b_1b_2)\\ &= y_1a_1a_2\\ &= y_1a_1\cdot(a_2,b_2)\\ &= (y_1\cdot(a_1,b_1))\cdot(a_2,b_2);\end{aligned}$$

and

$$\begin{aligned}y_1\cdot(1,1) &= y_11\\ &= y_1.\end{aligned}$$

Thus, $p_A(D)$ is a module over D.

2. (a) We will show that i is a module homomorphism over D. Let $x_1,x_2\in Ker(\beta)$ and $(a,b)\in D$. Then

$$\begin{aligned}i(x_1+x_2) &= (0,x_1+x_2)\\ &= (0,x_1)+(0,x_2)\\ &= i(x_1)+i(x_2)\end{aligned}$$

and

$$\begin{aligned} i((a,b)\cdot x_1) &= i(bx_1) \\ &= (0,bx_1) \\ &= (a,b)\cdot(0,x_1) \\ &= (a,b)\cdot i(x_1). \end{aligned}$$

(b) We will show that p_A is a module homomorphism over D. Let $(a,b),(y,x)\in D$. Then

$$\begin{aligned} p_A((a,b)+(y,x)) &= p_A((a+y,b+x)) \\ &= a+y \\ &= p_A(a,b)+p_A(y,x) \end{aligned}$$

and

$$\begin{aligned} p_A((a,b)\cdot(y,x)) &= p_A(ay,bx) \\ &= ay \\ &= (a,b)\cdot y \\ &= (a,b)\cdot p_A(y,x). \end{aligned}$$

3. (a) We will show that i is injective. Let $x\in Ker(\beta)$. Then

$$\begin{aligned} Ker(i) &= \{x\in Ker(\beta)\mid i(x)=0_D\} \\ &= \{x\in Ker(\beta)\mid (0,x)=(0,0)\} \\ &= \{x\in Ker(\beta)\mid x=0\} \\ &= \{0\}. \end{aligned}$$

(b) We will show that $Im(i)=Ker(p_A)$. Let $(a,b)\in D$. Then

$$\begin{aligned} Im(i) &= \{(a,b)\in D\mid i(x)=(a,b),\ \text{for some } x\in Ker(\beta)\} \\ &= \{(a,b)\in D\mid (0,x)=(a,b)\} \\ &= \{(a,b)\in D\mid a=0,b=x\} \\ &= \{(0,b)\} \end{aligned}$$

and

$$Ker(p_A)=\{(a,b)\in D\mid p_A(a,b)=0_A\}=\{(a,b)\in D\mid a=0\}=\{(0,b)\}.$$

(c) Let $y\in A$. We will prove that there exists $(a,b)\in D$ such that $p_A((a,b))=y$. Note that based on the definition of p_A we obtain $p_A((y,b))=y$ for any $b\in B$. It means that for every $y\in A$ there is $(a,b)\in D$, namely $(a,b)=(y,b)$ such that $p_A((a,b))=y$. Thus, p_A is surjective.

Based on 1, 2, 3, it is proven that the sequence $0\to Ker(\beta)\xrightarrow{i} D\xrightarrow{p_A} p_A(D)\to 0$ is an exact sequence of module over D. Thus, by Proposition 1.2 we get D is a Noetherian module over D (D is a Noetherian ring) if and only if $Ker(\beta)$ and $p_A(D)$ are Noetherian modules over D (Noetherian rings). □

We can characterize D as an Artinian ring which is explained below.

Proposition 3.4. With notation from Definition 3.1, $D := \alpha \times_C \beta$ is an Artinian ring if only if $Ker(\beta)$ and $p_A(D)$ are Artinian rings (Artinian modules over D).

Proof. By Proposition 1.3, we prove that $0 \to Ker(\beta) \xrightarrow{i} D \xrightarrow{p_A} p_A(D) \to 0$ is exact sequence. It is clear from Proposition 3.3 that the sequence is an exact sequence of modules over D. □

4. Properties of Amalgamated Algebras Along an Ideal

Since amalgamated algebras along an ideal is a ring, it has ideals. Furthermore, we can form quotient rings from those ideals and identify isomorphisms using The First Fundamental Theorem of Ring Homomorphism.

We take several points from the proposition in the previous paper [1] that are related to our research purposes.

Proposition 4.1. ([1]*Proposition* 5.1) Let $f : A \to B$ be a ring homomorphism, J be an ideal of B, and $A \bowtie^f J = \{(a, f(a) + j) \mid a \in A, j \in J\}$.

1. Let $i : A \to A \bowtie^f J$ be a ring homomorphism defined by $i(a) := (a, f(a))$ for all $a \in A$. Then A can be embedded in $A \bowtie^f J$.
2. Let $p_A : A \bowtie^f J \to A$ and $p_B : A \bowtie^f J \to B$. Then
   (a) $\dfrac{A \bowtie^f J}{(\{0\} \times J)}$ and A are isomorphic.
   (b) $\dfrac{A \bowtie^f J}{f^{-1}(J) \times \{0\}}$ and $f(A) + J$ are isomorphic.

Proof.

1. Let $a \in A$. Then
$$Ker(i) = \{a \in A \mid i(a) = (0, 0)\} = \{a \in A \mid (a, f(a)) = (0, 0)\} = \{0\}.$$
   Thus, A can be embedded in $A \bowtie^f J$.
2. We will prove that p_A is surjective, where $p_A : A \bowtie^f J \to A$ is defined by $(a, f(a) + j) \mapsto a$. Let $x \in A$. We will prove that there exists $(a, f(a) + j) \in A \bowtie^f J$ such that $p_A((a, f(a) + j)) = x$. Note that based on definition we obtain $p_A((x, f(x)+j)) = x$ for any $f(x)+j \in B$. It means that for every $x \in A$ there exists $(a, f(a) + j) \in A \bowtie^f J$, namely $(a, f(a) + j) = (x, f(x) + j)$ such that $p_A((a, f(a) + j)) = x$. Therefore, p_A is surjective. Next, we will prove that $Ker(p_A) = \{0\} \times J$. Let $(a, f(a) + j) \in A \bowtie^f J$. Then
$$\begin{aligned} Ker(p_A) &= \{(a, f(a) + j) \in A \bowtie^f J \mid p_A((a, f(a) + j)) = 0_A\} \\ &= \{(a, f(a) + j) \in A \bowtie^f J \mid a = 0_A\} \\ &= \{(0, f(0) + j) \in A \bowtie^f J\} \\ &= \{(0, j) \in A \bowtie^f J\} \\ &= \{0\} \times J. \end{aligned}$$
   Furthermore, we will prove that $p_B(A \bowtie^f J) = f(A) + J$.
$$\begin{aligned} p_B(A \bowtie^f J) &= \{p_B((a, f(a) + j)) \in B \mid a \in A, j \in J\} \\ &= \{f(a) + j \in B \mid a \in A, j \in J\} \\ &= f(A) + J. \end{aligned}$$

Next we will prove that $Ker(p_B) = f^{-1}(J) \times \{0\}$. Let $(a, f(a) + j) \in A \bowtie^f J$.

$$\begin{aligned} Ker(p_B) &= \{(a, f(a) + j) \in A \bowtie^f J \mid p_B((a, f(a) + j)) = 0_B\} \\ &= \{(a, f(a) + j) \in A \bowtie^f J \mid f(a) + j = 0_B\} \\ &= \{(a, 0) \in A \bowtie^f J \mid f(a) = -j = k \text{ for some } k \in J\} \\ &= \{(a, 0) \in A \bowtie^f J \mid f(a) \in J\} \\ &= f^{-1}(J) \times \{0\}. \end{aligned}$$

Therefore, using the First Fundamental Theorem of Ring Homomorphism, we have:

(a) $\dfrac{A \bowtie^f J}{(\{0\} \times J)}$ and A are isomorphic.

(b) $\dfrac{A \bowtie^f J}{f^{-1}(J) \times \{0\}}$ and $f(A) + J$ are isomorphic.

□

5. Characterization of Amalgamated Algebras Along an Ideal as Prime Rings, Reduced Rings, Noetherian Rings, and Artinian Rings

Based on the definition, construction, and properties of amalgamated algebras along an ideal, we can characterize amalgamated algebras along an ideal as prime rings, reduced rings, Noetherian rings, and Artinian rings.

Proposition 5.1. ([4]*Theorem* 2.8) With notation from Proposition 4.1, $A \bowtie^f J$ is a prime ring if only if $f(A) + J$ is a prime ring and $f^{-1}(J) = \{0\}$.

Proof. ($\Rightarrow$) First, we will prove that $f^{-1}(J) = \{0\}$. Let $A \bowtie^f J$ be a prime ring. Based on Proposition 4.1 2, it is known that in the natural projections $p_A : A \bowtie^f J \to A$ and $p_B : A \bowtie^f J \to B$ we have $Ker(p_A) = \{0\} \times J$ and $Ker(p_B) = f^{-1}(J) \times \{0\}$. Thus, $\{0\} \times J$ and $f^{-1}(J) \times \{0\}$ are ideals in $A \bowtie^f J$. Since $A \bowtie^f J$ is a prime ring, $(\{0\} \times J)(f^{-1}(J) \times \{0\}) = \{(0, 0)\}$ implies $\{0\} \times J = \{(0, 0)\}$ or $f^{-1}(J) \times \{0\} = \{(0, 0)\}$. Since $J \neq \{0\}$, we obtain $f^{-1}(J) \times \{0\} = \{(0, 0)\}$. Thus, it is proven that $f^{-1}(J) = \{0\}$. Next, we will prove that $f(A) + J$ is a prime ring. Based on Proposition 4.1 2, $\dfrac{A \bowtie^f J}{f^{-1}(J) \times \{0\}} \cong f(A)+J$. Therefore, we get $A \bowtie^f J+\{(0, 0)\} \cong f(A)+J$, equivalently, $A \bowtie^f J \cong f(A)+J$. Since $A \bowtie^f J$ is a prime ring, $f(A) + J$ is proven to be a prime ring (based on the structural property of isomorphism).
($\Leftarrow$) Since $f^{-1}(J) = \{0\}$ and based on Proposition 4.1 2, we get $A \bowtie^f J + \{(0, 0)\} \cong f(A) + J$. Equivalently, $A \bowtie^f J \cong f(A) + J$. Since $f(A) + J$ is a prime ring, $A \bowtie^f J$ is also a prime ring (based on the structural property of isomorphism). □

Proposition 5.2. With notation from Proposition 4.1, $A \bowtie^f J$ is a reduced ring if only if B is a reduced ring and $Nil(A) \cap f^{-1}(J) = \{0\}$.

Proof. ($\Rightarrow$) Let $A \bowtie^f J$ be a reduced ring. We will prove that $Nil(A) \cap f^{-1}(J) = \{0\}$. Note that, $f^{-1}(J) = Ker(\breve{f})$. Then

$$\begin{aligned} Ker(\breve{f}) &= \{a \in A \mid \breve{f}(a) = 0 + J\} \\ &= \{a \in A \mid (\pi \circ f)(a) = 0 + J\} \\ &= \{a \in A \mid \pi(f(a)) = 0 + J\} \\ &= \{a \in A \mid f(a) + J = 0 + J\} \\ &= \{a \in A \mid f(a) \in J\} \\ &= \{a \in f^{-1}(J)\} \\ &= f^{-1}(J). \end{aligned}$$

Let $a \in Nil(A) \cap Ker(\breve{f})$. It means that $a \in Nil(A)$ and $a \in Ker(\breve{f})$. Since $a \in Nil(A)$, it follows that $a^k = 0_A$ for some $k \in \mathbb{N}$. Since $a \in Ker(\breve{f})$, $\breve{f}(a) = 0 + J$. We get $(a, 0) \in Nil(A \bowtie^f J)$ because $(a, 0)^k = (a^k, 0^k) = (0, 0)$ and satisfies $\breve{f}(a) = 0 + J = \pi(0)$. Since $A \bowtie^f J$ is a reduced ring, the only nilpotent element in $A \bowtie^f J$ is $(0, 0)$. We obtain $a = 0$. Therefore, $Nil(A) \cap f^{-1}(J) = \{0\}$. Next, we will prove that B is a reduced ring. Since $A \bowtie^f J$ is a reduced ring, $(0, 0)$ is an unique nilpotent element in $A \bowtie^f J$. Suppose that $j \in Nil(B)$. We have $j^p = 0$ for some $p \in \mathbb{N}$. Then, we get $(0, j)$ is nilpotent element of $A \bowtie^f J$ because $(0, j)^p = (0^p, j^p) = (0, 0)$ and satisfies $\breve{f}(0) = \pi(f(0)) = 0 + J = j + J = \pi(j)$, contradiction to $A \bowtie^f J$ as a reduced ring. So, we obain $j = 0$. Therefore, B is a reduced ring.
($\Leftarrow$) Let B be a reduced ring and $Nil(A) \cap f^{-1}(J) = \{0\}$. Note that, $f^{-1}(J) = Ker(\breve{f})$. We shall prove that $A \bowtie^f J$ is a reduced ring. Let $(a, b) \in Nil(A \bowtie^f J)$. It means that $(a, b)^r = (0, 0)$ for some $r \in \mathbb{N}$. We obtain $b^r = 0$. Then $b \in Nil(B)$. Since B is reduced ring, we have $b = 0$. So $(a, b) = (a, 0) \in Nil(A \bowtie^f J)$. We get $a \in Nil(A)$ and satisfies $\breve{f}(a) = \pi(f(a)) = f(a) + J = 0 + J = \pi(0)$. It implies that $a \in Ker(f)$. We have $a \in Nil(A) \cap Ker(\breve{f})$. Because $Nil(A) \cap Ker(\breve{f}) = \{0\}$, $a = 0$. Consequently, $(a, 0) = (0, 0) \in Nil(A \bowtie^f J)$. Thus, the only nilpotent element in $A \bowtie^f J$ is $(0, 0)$. Therefore, $A \bowtie^f J$ is a reduced ring. □

Now, we characterize amalgamated algebras along an ideal as a Noetherian ring.

Proposition 5.3. ([1]*Proposition* 5.6) With notation from Proposition 4.1 , $A \bowtie^f J$ is a Noetherian ring if only if A and $f(A) + J$ are Noetherian rings.

Proof. ($\Rightarrow$) Since $A \bowtie^f J$ is a Noetherian ring and there exist natural homomorphism rings,

$$\pi_1 : (A \bowtie^f J) \to \frac{A \bowtie^f J}{(\{0\} \times J)} \text{ and } \pi_2 : (A \bowtie^f J) \to \frac{A \bowtie^f J}{f^{-1}(J) \times \{0\}}$$

are surjective homomorphisms. We obtain that those quotient rings are Noetherian rings. Based on Proposition 4.1 2, A and $f(A) + J$ are Noetherian rings.
($\Leftarrow$) Let A and $f(A)+J$ be Noetherian rings. We will prove that $A \bowtie^f J$ is a Noetherian ring. Based on Proposition 3.2, $A \bowtie^f J$ is equivalent to *"pullback"* or *"fiber product"* $\breve{f} \times_{B/J} \pi$. Applying Proposition 3.3, we can form this sequence

$$0 \to Ker(\pi) \xrightarrow{i} A \bowtie^f J \xrightarrow{p_A} p_A(A \bowtie^f J) \to 0.$$

Note that, $Ker(\pi) = \{b \in B \mid \pi(b) = 0 + J\} = \{b \in B \mid b + J = 0 + J\} = \{b \in B \mid b \in J\} = J$ and $p_A(A \bowtie^f J) = A$. We obtain:

$$0 \to J \xrightarrow{i} A \bowtie^f J \xrightarrow{p_A} A \to 0.$$

Note that, J can be viewed as a module over $A \bowtie^f J$ because p_B induces J as module over $A \bowtie^f J$ with $(a, f(a) + j) \cdot j_1 := p_B(a, f(a) + j)j_1$ for all $a \in A$ and $j, j_1 \in J$. Therefore, J is a Noetherian ring because J as a submodule over $A \bowtie^f J$ is an ideal of Noetherian ring $f(A) + J$. By applying Proposition 3.3, because A and J are Noetherian rings, we have $A \bowtie^f J$ is a Noetherian ring. □

From previous proposition, we try to investigate characterization of amalgamated algebras along an ideal as an Artinian ring which was not discussed in previous reference paper.

Proposition 5.4. With notation from Proposition 4.1, $A \bowtie^f J$ is an Artinian ring if only if A and $f(A) + J$ are Artinian rings, where every ideal of J has unity.

Proof. (⇒) This proof is not much different with the previous proposition. Since $A \bowtie^f J$ is an Artinian ring, the quotient rings $\dfrac{A \bowtie^f J}{(\{0\} \times J)}$ and $\dfrac{A \bowtie^f J}{f^{-1}(J) \times \{0\}}$ are Artinian rings. Based on Proposition 4.1 2, A and $f(A) + J$ are Artinian rings.
(⇐) Let A and $f(A) + J$ be Artinian rings. Based on the Proposition 3.3, we can form the sequence

$$0 \longrightarrow Ker(\pi) \xrightarrow{i} A \bowtie^f J \xrightarrow{p_A} p_A(A \bowtie^f J) \longrightarrow 0.$$

It is clear from Proposition 5.3 that the sequence above become the following exact sequence:

$$0 \longrightarrow J \xrightarrow{i} A \bowtie^f J \xrightarrow{p_A} A \longrightarrow 0.$$

Note that, J is an ideal of B, so, J is an ideal of $f(A) + J$. Let I be an ideal of J. Since J is an ideal of $f(A) + J$ and every ideal of J has unity, I is an ideal of $f(A) + J$ from Proposition 1.4. It is known that $f(A) + J$ is an Artinian ring. Thus, J is an Artinian ring. By applying the Proposition 3.4, since A and J are Artinian rings we have $A \bowtie^f J$ is an Artinian ring. □

6. Conclusion

Based on definition, construction, and properties of amalgamated algebras along an ideal we can characterize it as prime rings, reduced rings, Noetherian rings, and Artinian rings. Especially for reduced rings and artinian rings, $A \bowtie^f J$ is a reduced ring if only if B is a reduced ring and $Nil(A) \cap f^{-1}(J) = \{0\}$. We also have that $A \bowtie^f J$ is an Artinian ring if only if A and $f(A) + J$ are Artinian rings, where every ideal of J has unity.

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