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Analysis of Topological Indices in Unit Graphs of Modular Integer Rings

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A B S T R A C T

Topological indices are numerical graph invariants that reflect structural properties of graphs and have broad applications in chemistry, algebra, and network analysis. This paper focuses on the analysis of several topological indices in the context of unit graphs associated with modular integer rings. In a unit graph, vertices represent ring elements, and two vertices are adjacent if their sum is a unit. We investigate and derive general formulas for six indices: the Narumi-Katayama index, the Forgotten index, the Atom-Bond Connectivity (ABC) index, the first and second Gourava indices, and the first Revan index. Two cases are considered for the ring of integers modulo n, namely when n is a power of 2 and when n is an odd prime. The results offer a deeper understanding of the algebraic and combinatorial properties of unit graphs and contribute to the development of algebraic graph theory.

Keywords: Narumi-Katayama index, Forgotten index, ABC index, Gourava index, Revan index

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1. Introduction

Graph theory was first formally introduced by Leonhard Euler in 1736 through his seminal work Solutio Problematis ad Geometriam Situs Pertinentis, in which he solved the classical Königsberg Bridge Problem [1]. This work marked the birth of graph theory as a branch of mathematics that studies the relationships between objects through vertices and edges. Decades later, in 1857, Arthur Cayley developed the concept of trees to represent molecular structures in organic chemistry, which later became the foundation for the application of graph theory in computational and mathematical chemistry [2].

Over time, graph theory has undergone significant expansion, extending beyond chemistry and computer science to encompass algebraic structures, particularly groups and rings. One interesting approach in this field is the representation of rings through their associated unit graphs. The concept of the unit graph was first introduced by Ashrafi, et al. (2010) in their paper "Unit Graphs Associated with Rings." A unit graph is defined as a graph whose vertex set corresponds to the elements of a ring, where two distinct vertices u and v are said to be adjacent if and only if $u + v$ is a unit in the ring [3].

More recently, Lestari et al. (2024) investigated the structural properties of unit graphs over the ring of integers modulo n and successfully characterized their structure along with several important graph invariants such as vertex degree, graph diameter, and chromatic number [4]. Their findings revealed that the connectivity pattern of the unit graph depends on the factorization of n, when n is a power of two, the unit graph forms a complete bipartite graph, whereas when n is an odd prime, the resulting graph is complete multipartite. These results provide an essential foundation for understanding the relationship between the algebraic properties of the ring $\mathbb{Z}_n$ and the combinatorial structure of its corresponding graph.

In addition, several other studies have explored topological indices on various algebraic graphs. For instance, Asmarani, et al. (2023) examined several topological indices of power graphs constructed from dihedral groups and established relationships between group structures and index values [5]. Husni, et al. (2022) analyzed coprime graphs of the integer modulo group with prime power order and derived general expressions for the harmonic and Gutman indices [6]. Meanwhile, Malik, et al. (2024) studied nilpotent graphs defined on the ring of integers modulo a prime power and linked them to several chemical topological indices to uncover the algebraic characteristics of such graphs [7].

Building upon these studies, the present work aims to extend the analysis of unit graphs over the ring of integers modulo n by focusing on the determination and formulation of several topological indices that quantitatively represent the structural characteristics of these graphs. Topological indices serve as numerical invariants that bridge algebraic and graphical representations, allowing complex structures to be interpreted quantitatively. Therefore, this study focuses on deriving and analyzing several important indices, including the Narumi–Katayama index, the Forgotten index, the Atom–Bond Connectivity (ABC) index, the first and second Gourava indices, and the first Revan index, which are expected to contribute to a deeper understanding of the topological and combinatorial properties of unit graphs in algebraic graph theory.

2. Result and Discussion

To support the analysis of the topological indices presented, it is essential to first understand the basic structure of the unit graph used as the subject of study. Therefore, the formal definition of the unit graph in a ring is presented below.

Definition 2.1. [4] The unit graph of R, denoted by $G(R)$, has its set of vertices equal to the set of all elements of R, distinct vertices u and v are adjacent if and only if $u + v$ is a unit of R.

To illustrate this definition, we provide the following example of a unit graph constructed on a specific ring of integers modulo n

Example 2.1. Consider the unit graph of $\mathbb{Z}_n$ for $n = 2^2$. We have $\mathbb{Z}_4 = \{0, 1, 2, 3\}$. The units of $\mathbb{Z}_4$ are $U(\mathbb{Z}_4) = \{1, 3\}$. Two vertices a and b in $\mathbb{Z}_4$ are adjacent if and only if $a + b \in U(\mathbb{Z}_4)$. Thus, the set of adjacent vertices is $\{(0, 1), (0, 3), (1, 2), (2, 3)\}$.

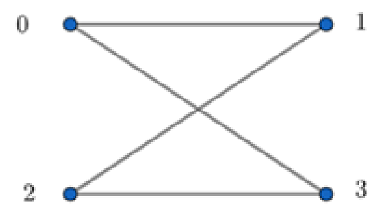

Figure 1. Unit Graph for $\mathbb{Z}_4$

Example 2.2. Consider the unit graph of $\mathbb{Z}_n$ for $n = 5$. We have $\mathbb{Z}_5 = \{0, 1, 2, 3, 4\}$. The units of $\mathbb{Z}_5$ are $U(\mathbb{Z}_5) = \{1, 2, 3, 4\}$. Two vertices a and b in $\mathbb{Z}_5$ are adjacent if and only if $a + b \in U(\mathbb{Z}_5)$ is a unit. Thus, the set of adjacent vertices is $\{(0, 1), (0, 2), (0, 3), (0, 4), (1, 2), (1, 3), (4, 2), (4, 3)\}$.

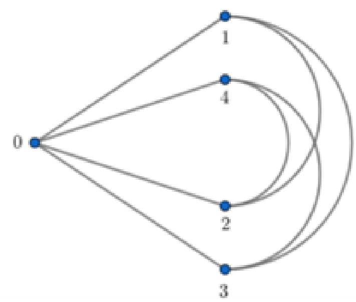

Figure 2. Unit Graph for $\mathbb{Z}_5$

Each vertex in the unit graph $G(R)$ represents an element of R, and the relationships between these vertices are determined by the additive properties of the elements that are units, which in turn affect the degree of each vertex in the graph. The degree of the graph is defined below.

Definition 2.2. [6] The degree of a graph is the number of edges that are incident to the vertex. It is annotated as d_u for any vertices.

With the vertex degree defined, we can now introduce the Narumi-Katayama index as a fundamental topological measure derived from vertex degrees.

Definition 2.3. [8] Suppose G is a connected graph, where the set of vertices is denoted by $V(G)$ and the set of edges by $E(G)$. The Narumi-Katayama index is introduced as follows:

$$NK(G) = \prod_{u \in V(G)} d_u.$$

Example 2.3. Based on Example 2.1 and Definition 2.3, the Narumi-Katayama index is calculated as follows:

$$\begin{aligned} NK(G(\mathbb{Z}_4)) = &\prod_{u \in V(G(\mathbb{Z}_4))} d_u \\ =& d_0 \times d_1 \times d_2 \times d_3 \\ =& 2 \times 2 \times 2 \times 2 \\ =& 16. \end{aligned}$$

The Forgotten Index, commonly referred to as the F-index, defined as the sum of the cubes of vertex degrees in a graph, the F-index is expressed mathematically as,

Definition 2.4. [9] Suppose G is a connected graph, where the set of vertices is denoted by $V(G)$ and the set of edges by $E(G)$. The Forgotten index is introduced as follows:

$$F(G) = \sum_{u \in V(G)} (d_u)^3.$$

Example 2.4. Based on Example 2.2 and Definition 2.4, the Forgotten index is calculated as follows:

$$\begin{aligned} F(G(\mathbb{Z}_5)) = &\sum_{u \in V(G)} d_u^3 \\ =& d_0^3 + d_1^3 + d_2^3 + d_3^3 + d_4^3 \\ =& 4^3 + 3^3 + 3^3 + 3^3 + 3^3 \\ =& 4^3 + 4 \cdot 3^3 \\ =& 172. \end{aligned}$$

After discussing the Narumi-Katayama and Forgotten indices, we now turn our attention to another important topological descriptor frequently used in chemical graph theory, namely the Atom-Bond Connectivity (ABC) index.

Definition 2.5. [10] Suppose G is a connected graph, where the set of vertices is denoted by $V(G)$ and the set of edges by $E(G)$. The Atom-Bond Connectivity index is introduced as follows:

$$ABC(G) = \sum_{uv \in E(G)} \sqrt{\frac{d_u + d_v - 2}{d_u d_v}}.$$

Example 2.5. Based on Example 2.1 and Definition 2.5, the ABC index is calculated as follows:

$$\begin{aligned} ABC(G(\mathbb{Z}_4)) = &\sum_{uv \in E(G(\mathbb{Z}_2))} \sqrt{\frac{d_u + d_v - 2}{d_u d_v}} \\ =& \sqrt{\frac{2+2-2}{2 \times 2}} + \sqrt{\frac{2+2-2}{2 \times 2}} + \sqrt{\frac{2+2-2}{2 \times 2}} + \sqrt{\frac{2+2-2}{2 \times 2}} \\ =& 4\sqrt{\frac{2}{4}} \\ =& 2\sqrt{2}. \end{aligned}$$

The ABC index provides valuable insights into molecular stability and branching patterns within a graph, particularly in applications related to chemistry. However, to further explore vertex degree interactions in a more generalized combinatorial context, other indices have been introduced. Among them are the first and second Gourava indices, which consider additive and multiplicative combinations of vertex degrees across all edges. These indices allow for a more detailed characterization of the structure and complexity of a graph.

Definition 2.6. [11] The first and second Gourava indices of the graph are denoted by $GO_1(G)$ and $GO_2(G)$, and are defined as follows:

$$GO_1(G) = \sum_{uv \in E(G)} [(d_u + d_v) + d_u d_v]$$

$$GO_2(G) = \sum_{uv \in E(G)} \left(d_u^2 d_v + d_u d_v^2 \right).$$

Example 2.6. Based on Example 2.2 and Definition 2.6, the First and Second Gourava index as follow,

$$\begin{aligned} GO_1(G(\mathbb{Z}_5)) = & \sum_{uv \in E(G(\mathbb{Z}_5))} [(d_u + d_v) + d_u d_v] \\ = & [(4+3) + 4 \times 3] + [(4+3) + 4 \times 3] + [(4+3) + 4 \times 3] + [(4+3) + 4 \times 3] \\ & + [(3+3) + 3 \times 3] + [(3+3) + 3 \times 3] + [(3+3) + 3 \times 3] + [(3+3) + 3 \times 3] \\ = & 4(7+12) + 4(6+9) \\ = & 136. \end{aligned}$$

$$\begin{aligned} GO_2(G(\mathbb{Z}_5)) = & \sum_{uv \in E(G(\mathbb{Z}_5))} \left(d_u^2 d_v + d_u d_v^2 \right) \\ = & (4^2 \times 3 + 4 \times 3^2) + (4^2 \times 3 + 4 \times 3^2) + (4^2 \times 3 + 4 \times 3^2) + (4^2 \times 3 + 4 \times 3^2) \\ & + (3^2 \times 3 + 3 \times 3^2) + (3^2 \times 3 + 3 \times 3^2) + (3^2 \times 3 + 3 \times 3^2) + (3^2 \times 3 + 3 \times 3^2) \\ = & 4(4^2 \times 3 + 4 \times 3^2) + 4(3^2 \times 3 + 3 \times 3^2) \\ = & 4 \times 84 + 4 \times 54 \\ = & 552. \end{aligned}$$

While the Gourava indices incorporate both additive and multiplicative aspects of vertex degrees, further refinement in vertex-based topological analysis can be achieved through the Revan index. This index introduces a degree-based transformation by incorporating the maximum and minimum degrees in the graph, offering a complementary perspective on graph structure that emphasizes deviations from extremal degree values.

Definition 2.7. [12] Let $\Delta(G)(\delta(G))$ denote the maximum (minimum) degree among the vertices of G. The Revan vertex degree of a vertex in G is defined $r_u = \Delta(G) + \delta(G) - d_u$. The First Revan index of the graph is denoted by $R_1(G)$ defined as,

$$R_1(G) = \sum_{uv \in E(G)} (r_u + r_v).$$

Example 2.7. Based on Example 2.1 and Definition 2.7, $\Delta(G(\mathbb{Z}_4)) = 2$ and $\delta(G(\mathbb{Z}_4)) = 2$.
So, the First Revan index as follow

$$\begin{aligned} R_1(G) = & \sum_{uv \in E(G)} (r_u + r_v) \\ = & (r_0 + r_1) + (r_0 + r_3) + (r_2 + r_1) + (r_2 + r_3) \\ = & (2+2) + (2+2) + (2+2) + (2+2) \\ = & 4(2+2) \\ = & 16. \end{aligned}$$

After presenting examples of each topological index, we now shift our focus to a more general theoretical framework. In particular, we derive closed-form expressions for these indices by considering specific algebraic properties of the ring $\mathbb{Z}_n$. The following lemmas describe the degree patterns of unit graphs when n is either a power of two or an odd prime, which serve as the foundation for the subsequent theorems.

Lemma 2.1. *[4] Suppose $\mathbb{Z}_n$ be an integer modulo ring of order $n = 2^k$, where $k \in \mathbb{N}$. Then, the unit graph $G(\mathbb{Z}_n)$ possesses a numerical invariant, where the degree of each vertex u is given by $d_u = \frac{n}{2}$ for all u in the vertex set $V(G(\mathbb{Z}_n))$.*

Lemma 2.2. *[4] Suppose $\mathbb{Z}_n$ be an integer modulo ring of order n where n is an odd prime number. Then, the unit graph $G(\mathbb{Z}_n)$ has $d_0 = n - 1$, and $d_u = n - 2$, for $\forall u \in V(G(\mathbb{Z}_n)) \setminus \{0\}$.*

With the vertex degrees characterized for both even-power and odd-prime modulus cases, we are now equipped to formulate explicit expressions for the topological indices under these structural constraints. The following theorems utilize the established degree patterns to derive closed-form formulas for each index, starting with the Narumi-Katayama index.

Theorem 2.3. *Let $G(\mathbb{Z}_n)$ be the unit graph of the integer ring modulo n, then the Narumi-Katayama index of $G(\mathbb{Z}_n)$ is*

(i) for $n = 2^k$, $k \in \mathbb{N}$: $NK(G(\mathbb{Z}_n)) = n \cdot \left(\frac{n}{2}\right)^3$
(ii) for n is an odd prime number : $NK(G(\mathbb{Z}_n)) = (n-1)\left((n-1)^2 + (n-2)^3\right)$.

Proof. (i) By using Lemma 2.1, the Narumi-Katayama index of unit graph for $\mathbb{Z}_{2^k}$,

$$\begin{aligned} NK(G(\mathbb{Z}_n)) &= \prod_{u \in V(G(\mathbb{Z}_n))} d_u \\ &= \underbrace{\left(\frac{n}{2}\right) \times \left(\frac{n}{2}\right) \times \ldots \times \left(\frac{n}{2}\right)}_{n} \\ &= \left(\frac{n}{2}\right)^n \end{aligned}$$

(ii) By using Lemma 2.2, the Narumi-Katayama index of unit graph for $\mathbb{Z}_n$, where n is an odd prime number,

$$\begin{aligned} NK(G(\mathbb{Z}_n)) &= \prod_{u \in V(G(\mathbb{Z}_n))} d_u \\ &= d_0 \times \prod_{u \in V(G(\mathbb{Z}_n)) \setminus \{0\}} d_u \\ &= (n-1) \times \underbrace{(n-2) \times (n-2) \times \ldots \times (n-2)}_{n-1} \\ &= (n-1)(n-2)^{n-1}. \end{aligned}$$

□

After explaining the degree of each vertex in the unit graph for two modulus cases when the modulus is a power of two and when it is an odd prime number Theorem 2.3 presents a general formula for the Narumi-Katayama index. This index is a multiplicative topological invariant that depends on the degrees of the vertices. Following this, Theorem 2.4 introduces the Forgotten index, an invariant calculated based on the sum of the cubes of the vertex degrees. Unlike the previous index, which involves multiplication, this one uses addition, offering a different perspective on the structure and connectivity of the unit graph. Together, these theorems enhance the understanding of how the algebraic structure of the ring of integers modulo a certain number influences its graphical representation.

Theorem 2.4. *Let $G(\mathbb{Z}_n)$ be the unit graph of the integer ring modulo n, then the Forgotten index of $G(\mathbb{Z}_n)$ is*

(i) for $n = 2^k$, $k \in \mathbb{N}$: $F(G(\mathbb{Z}_n)) = n \cdot \left(\frac{n}{2}\right)^3$.
(ii) for n is an odd prime number : $F(G(\mathbb{Z}_n)) = (n-1)\left((n-1)^2 + (n-2)^3\right)$.

Proof. (i) By using Lemma 2.1, each vertex of the graph has degree $\frac{n}{2}$. So, we get

$$\begin{aligned} F(G(\mathbb{Z}_n)) &= \sum_{u \in V(G(\mathbb{Z}_n))} d_u^3 \\ &= \underbrace{\left(\frac{n}{2}\right)^3 + \left(\frac{n}{2}\right)^3 + \cdots + \left(\frac{n}{2}\right)^3}_{n} \\ &= n \cdot \left(\frac{n}{2}\right)^3. \end{aligned}$$

(ii) By using Lemma 2.2, the Forgotten index of unit graph for $\mathbb{Z}_n$, where n is odd prime number,

$$\begin{aligned} F(G(\mathbb{Z}_n)) &= \sum_{u \in V(G(\mathbb{Z}_n))} d_u^3 \\ &= (n-1)^3 + \underbrace{(n-2)^3 + (n-2)^3 + \cdots + (n-2)^3}_{n-1} \\ &= (n-1)^3 + (n-1)(n-2)^3 \\ &= (n-1)\left((n-1)^2 + (n-2)^3\right). \end{aligned}$$

□

Theorem 2.4 concludes the discussion of the Forgotten index by providing explicit formulas for different modulus cases. This additive invariant complements the earlier multiplicative approach by capturing the distribution of vertex degrees in a different way. Building on this, Theorem 2.5 introduces the Atom-Bond Connectivity (ABC) index, a topological measure originally inspired by studies in chemistry. The ABC index reflects both additive and multiplicative interactions between adjacent vertices by considering their degrees in a square-root-based expression. This transition marks a shift toward more complex structural descriptors that incorporate pairwise relationships within the graph.

Theorem 2.5. *Let $G(\mathbb{Z}_n)$ be the unit graph of the integer ring modulo n, then the Atom-Bond Connectivity index of $G(\mathbb{Z}_n)$ is* *(i) for* $n = 2^k,\ k \in \mathbb{N}\ :\ ABC(G(\mathbb{Z}_n)) = \left(\frac{n}{2}\right)\sqrt{n-2}.$

(ii) for n is an odd prime number : $ABC(G(\mathbb{Z}_n)) = (n-1)\left(\sqrt{\frac{2n-5}{(n-1)(n-2)}} + \frac{(n-3)}{2(n-2)}\sqrt{2(n-3)}.\right)$

Proof. (i) By using Lemma 2.1, The Atom-Bond Connectivity index of unit graph for $\mathbb{Z}_n$, $n = 2^k$

$$
\begin{aligned}
ABC(G(\mathbb{Z}_n)) &= \sum_{uv \in E(G)} \sqrt{\frac{d_u + d_v - 2}{d_u d_v}} \\
&= \frac{\sqrt{\left(\frac{n}{2}\right) + \left(\frac{n}{2}\right) - 2}}{\left(\frac{n}{2}\right)\left(\frac{n}{2}\right)} + \frac{\sqrt{\left(\frac{n}{2}\right) + \left(\frac{n}{2}\right) - 2}}{\left(\frac{n}{2}\right)\left(\frac{n}{2}\right)} + \cdots + \frac{\sqrt{\left(\frac{n}{2}\right) + \left(\frac{n}{2}\right) - 2}}{\left(\frac{n}{2}\right)\left(\frac{n}{2}\right)} \\
&= \left(\frac{n}{2}\right)^2 \cdot \sqrt{\frac{\left(\frac{n}{2}\right) + \left(\frac{n}{2}\right) - 2}{\left(\frac{n}{2}\right)\left(\frac{n}{2}\right)}} \\
&= \frac{\left(\frac{n}{2}\right)^2}{\frac{n}{2}} \cdot \sqrt{n-2} \\
&= \frac{n}{2} \cdot \sqrt{n-2}.
\end{aligned}
$$

(ii) For n odd prime number: It is said that $G(\mathbb{Z}_n)$ is a complete bipartite graph with $\frac{n+1}{2}$ partition, so there are $V_1, V_2, \ldots, V_{\frac{n+1}{2}} \subseteq V(G(\mathbb{Z}_n))$, where all vertices are adjacent except for those that are in the same partition. Therefore, in this case, we will divide it into 2 cases to calculate the Atom-Bond Connectivity index,

- **Case 1.** For $\{0, v\} \subseteq V$ and $v \in G(\mathbb{Z}_n) \setminus \{0\}$, we obtain

$$
\begin{aligned}
&= \sum_{w \in E(G)} \sqrt{\frac{d_u + d_v - 2}{d_u d_v}} \\
&= \underbrace{\sqrt{\frac{(n-1)+(n-2)-2}{(n-1)(n-2)}} + \sqrt{\frac{(n-1)+(n-2)-2}{(n-1)(n-2)}} + \cdots + \sqrt{\frac{(n-1)+(n-2)-2}{(n-1)(n-2)}}}_{n-1} \\
&= (n-1)\sqrt{\frac{2n-5}{(n-1)(n-2)}}.
\end{aligned}
$$

- **Case 2.** For $u, v \in V(G(\mathbb{Z}_n)) \setminus \{0\}$, and for $u \in V_i$ dan $v \in V_j$, $i \neq j$, we obtain

$$
\begin{aligned}
&= \sum_{\substack{u \in V_i \\ v \in V_j \\ i \neq j}} \sqrt{\frac{d_u + d_v - 2}{d_u d_v}} \\
&= \underbrace{\sqrt{\frac{(n-2)+(n-2)-2}{(n-2)(n-2)}} + \sqrt{\frac{(n-2)+(n-2)-2}{(n-2)(n-2)}} + \cdots + \sqrt{\frac{(n-2)+(n-2)-2}{(n-2)(n-2)}}}_{\frac{(n-3)(n-1)}{2}}
\end{aligned}
$$

$$= \frac{(n-3)(n-1)}{2}\sqrt{\frac{2(n-3)}{(n-2)^2}}$$
$$= \frac{(n-3)(n-1)}{2(n-2)}\sqrt{2(n-3)}.$$

Thus, the ABC index of the unit graph of integers modulo n is the odd prime number,

$$ABC(G(\mathbb{Z}_n)) = (n-1)\sqrt{\frac{2n-5}{(n-1)(n-2)}} + \frac{(n-3)(n-1)}{2(n-2)}\sqrt{2(n-3)}$$
$$= (n-1)\left(\sqrt{\frac{2n-5}{(n-1)(n-2)}} + \frac{(n-3)}{2(n-2)}\sqrt{2(n-3)}\right).$$

□

After presenting the Atom-Bond Connectivity (ABC) index in Theorem 2.5, which takes into account interactions between vertex pairs through a function of their degrees, the discussion progresses to another class of degree-based topological measures. Theorem 2.6 introduces the First Gourava index, which combines both the sum and the product of degrees of adjacent vertices. This index provides a more comprehensive view of vertex interactions by simultaneously capturing their individual and joint degree characteristics. The shift from the ABC index to the Gourava index reflects a continued exploration of structural complexity within the unit graph.

Theorem 2.6. *Let $G(\mathbb{Z}_n)$ be the unit graph of the integer ring modulo n, then the First Gourava index of $G(\mathbb{Z}_n)$ is*

(i) for $n = 2^k$, $k \in \mathbb{N}$: $GO_1(G(\mathbb{Z}_n)) = \left(\frac{n}{2}\right)^3\left(\frac{n+4}{2}\right)$.
(ii) for n is an odd prime number : $GO_1(G(\mathbb{Z}_n)) = \frac{(n-3)(n-1)}{2}\left(2n^2-3n-1\right)$.

Proof. (i) For $G(\mathbb{Z}_n)$, where $n = 2^k$, by using Lemmma 2.1, we get the First Gourava Index

$$GO_1(G(\mathbb{Z}_n)) = \sum_{uv\in E(G)}[(d_u+d_v)+d_ud_v]$$
$$= \underbrace{\left[\left(\frac{n}{2}+\frac{n}{2}\right)+\left(\frac{n}{2}\right)\left(\frac{n}{2}\right)\right]+\ldots+\left[\left(\frac{n}{2}+\frac{n}{2}\right)+\left(\frac{n}{2}\right)\left(\frac{n}{2}\right)\right]}_{(\frac{n}{2})^2}$$
$$= \left(\frac{n}{2}\right)^2\left[\left(\frac{n}{2}+\frac{n}{2}\right)+\left(\frac{n}{2}\right)\left(\frac{n}{2}\right)\right]$$
$$= \left(\frac{n}{2}\right)^2\left[2\left(\frac{n}{2}\right)+\left(\frac{n}{2}\right)^2\right]$$
$$= \left(\frac{n}{2}\right)^3\left(2+\frac{n}{2}\right)$$
$$= \left(\frac{n}{2}\right)^3\left(\frac{n+4}{2}\right).$$

(ii) For n odd prime number, base on It is said that $G(\mathbb{Z}_n)$ is a complete bipartite graph with $\frac{n+1}{2}$ partition, so there are $V_1, V_2, \ldots, V_{\frac{n+1}{2}} \subseteq V(G(\mathbb{Z}_n))$, where all vertices are adjacent except for those that are in the same partition. Therefore, in this case, we will divide it into 2 cases to calculate the First Gourava index,

- **Case 1.** For $\{0, v\} \subseteq V$ and $v \in V(G(\mathbb{Z}_n)) \setminus \{0\}$, we obtain,

$$
\begin{aligned}
&= \sum_{0v \in E(G)} [(d_0 + d_v) + d_0 d_v] \\
&= \underbrace{[((n-1) + (n-2)) + (n-1)(n-2)] + \cdots + [((n-1) + (n-2)) + (n-1)(n-2)]}_{n-1} \\
&= (n-1)\,[((n-1) + (n-2)) + (n-1)(n-2)] \\
&= (n-1)(n^2 - n - 1).
\end{aligned}
$$

- **Case 2.** For $u, v \in V$ and $v \in V(G(\mathbb{Z}_n) \setminus \{0\}$ we obtain,

$$
\begin{aligned}
&= \sum_{\substack{u \in V_i \\ v \in V_j \\ i \neq j}} [(d_u + d_v) + d_u d_v] \\
&= \underbrace{[((n-2) + (n-2)) + (n-2)(n-2)] + \cdots + [((n-2) + (n-2)) + (n-2)(n-2)]}_{\frac{(n-3)(n-1)}{2}} \\
&= \frac{(n-3)(n-1)}{2}\left[2(n-2) + (n-2)^2\right] \\
&= \frac{(n-3)(n-1)(n-2)(n-1)}{2}.
\end{aligned}
$$

Thus, the First Gourava Index of the unit graph of integers modulo n is odd prime number

$$
\begin{aligned}
GO_1(G(\mathbb{Z}_n)) &= \frac{(n-3)(n-1)}{2}(n^2 - n - 1) + \frac{(n-3)(n-2)(n-1)n}{2} \\
&= \frac{(n-3)(n-1)}{2}\Big((n^2 - n - 1) + (n-2)n\Big) \\
&= \frac{(n-3)(n-1)}{2}(2n^2 - 3n - 1).
\end{aligned}
$$

□

Following the formulation of the First Gourava index in Theorem 2.6, which accounts for both additive and multiplicative combinations of adjacent vertex degrees, the analysis continues with the Second Gourava index in Theorem 2.7. This index extends the previous idea by focusing on squared degree terms, thereby emphasizing the influence of higher-degree vertices more strongly. The introduction of the Second Gourava index further enriches the topological characterization of the unit graph by offering a deeper measure of connectivity and degree variation across the graph structure.

Theorem 2.7. *Let $G(\mathbb{Z}_n)$ be the unit graph of the integer ring modulo n, then the Second Gourava index of $G(\mathbb{Z}_n)$ is*

(i) For $n = 2^k$, $k \in \mathbb{N}$: $GO_2(G(\mathbb{Z}_n)) = 2\left(\frac{n}{2}\right)^5$.
(ii) For n is an odd prime number : $GO_2(G(\mathbb{Z}_n)) = (n-1)^2(n-2)(2n-3) + (n-3)(n-2)^3(n-1)$.

Proof. (i) For $G(\mathbb{Z}_n)$, where $n = 2^k$, by using Lemma 2.2, we get the Second Gourava Index

$$
GO_2(G(\mathbb{Z}_n)) = \sum_{uv \in E(G)} \left(d_u^2 d_v + d_u d_v^2\right)
$$

$$= \underbrace{\left(\left(\frac{n}{2}\right)^2\left(\frac{n}{2}\right)+\left(\frac{n}{2}\right)\left(\frac{n}{2}\right)^2\right)+\cdots+\left(\left(\frac{n}{2}\right)^2\left(\frac{n}{2}\right)+\left(\frac{n}{2}\right)\left(\frac{n}{2}\right)^2\right)}_{(\frac{n}{2})^2}$$
$$= \left(\frac{n}{2}\right)^2\left(\left(\frac{n}{2}\right)^2\left(\frac{n}{2}\right)+\left(\frac{n}{2}\right)\left(\frac{n}{2}\right)^2\right)$$
$$= \left(\frac{n}{2}\right)^2\left(\left(\frac{n}{2}\right)^3+\left(\frac{n}{2}\right)^3\right)$$
$$= \left(\frac{n}{2}\right)^2\left(2\left(\frac{n}{2}\right)^3\right)$$
$$= 2\left(\frac{n}{2}\right)^5.$$

(ii) For n odd prime number: It is said that $G(\mathbb{Z}_n)$ is a complete bipartite graph with $\frac{n+1}{2}$ partition, so there are $V_1, V_2, \ldots, V_{\frac{n+1}{2}} \subseteq V(G(\mathbb{Z}_n))$, where all vertices are adjacent except for those that are in the same partition. Therefore, in this case, we will divide it into 2 cases to calculate the Second Gourava index.

- **Case 1.** For $\{0, v\} \subseteq V$ and $v \in V(G(\mathbb{Z}_n) \setminus \{0\})$, we obtain

$$\sum_{0v\in E(G)}\left(d_0^2 d_v + d_0 d_v^2\right) = \left((n-1)^2(n-2)+(n-1)(n-2)^2\right)+\cdots+\left((n-1)^2(n-2)+(n-1)(n-2)^2\right)$$
$$= (n-1)\left((n-1)^2(n-2)+(n-1)(n-2)^2\right)$$
$$= (n-1)^2(n-2)[(n-1)+(n-2)]$$
$$= (n-1)^2(n-2)(2n-3).$$

- **Case 2.** For $u, v \in V$ and $v \in V(G(\mathbb{Z}_n) \setminus \{0\})$, we obtain

$$\sum_{\substack{u\in V_i\\ v\in V_j\\ i\neq j}}\left(d_u^2 d_v + d_u d_v^2\right) = \left((n-2)^2(n-2)+(n-2)(n-2)^2\right)+\cdots+\left((n-2)^2(n-2)+(n-2)(n-2)^2\right)$$
$$= (n-3)(n-1)\left((n-2)^3+(n-2)^3\right)$$
$$= (n-3)(n-2)^3(n-1).$$

Thus, the Second Gourava index of the unit graph of the integers modulo n (with n odd prime number) is

$$GO_2(G(\mathbb{Z}_n)) = (n-1)^2(n-2)(2n-3)+(n-3)(n-2)^3(n-1).$$

□

3. Conclusions

In this article, the researchers successfully derived the general formulas for several topological indices of the unit graph. Various topological indices, including the Narumi-Katayama index, the Forgotten index, the Atom-Bond Connectivity index, the First and Second Gourava indices, as well as the First Revan index, have been formulated and analyzed for different cases. Several theorems have been established to determine these indices based on the degree of vertices in the unit graph. The findings of this study provide a mathematical framework for understanding the structural properties of unit graphs associated with integer rings modulo specific orders. These results contribute to the field of algebraic graph theory and have potential applications in computational algebra and network analysis. Future research may extend these concepts to more general ring structures or explore their applications in other areas of mathematics and science.

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