Volume 8 No 1 December (2025)

e-ISSN: 2615-3270
p-ISSN: 2615-3599

Eigen Mathematics Journal

homepage: http://eigen.unram.ac.id/index.php/eigen

Locating Chromatic Number for Rose Graphs and Barbell Operation

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ABSTRACT

The locating chromatic number of a graph is the minimum color required for a locating coloring. This concept is a combination of partition dimension and vertex coloring of a graph. The purpose of this paper is to determine the locating chromatic number of the Rose graph and the barbell Rose graphs. The method used to obtain the locating chromatic number of a graph is by determining its upper and lower bounds. In this paper, the locating chromatic number of the Rose graphs and its barbell operation were obtained. The locating chromatic number of Rose graph $M(C_n)$ is 4 for $n \in \{3, 4\}$ and 5 for $n \geq 5$. Furthermore, for barbell Rose graphs, 4 for $n = 3$ and 5 for $n \geq 4$.

Keywords: locating chromatic number; rose graph; barbell operation

Received : 08-07-2025;
Revised : 01-08-2025;
Accepted : 01-09-2025;
Published : 10-09-2025;

DOI: https://doi.org/10.29303/emj.v8i2.315

1. Introduction

Chartrand et al. [1] introduced locating chromatic numbers by merging two concepts in graph theory: vertex coloring and the partition dimension of a graph. The vertex coloring of a graph is a function that assigns a color to each vertex of such that no two adjacent vertices receive the same color [1]. The partition dimension is an extension of the metric dimension, which was first proposed by Chartrand et al. [2].

As research on graph invariants related to vertex identification progressed, attention also turned to other related parameters. In this context the discussion of locating chromatic numbers has been widely studied. Chartrand et al. [1] constructed a tree of order $n \geq 5$ with locating chromatic numbers varying from 3 to $(n - 1)$. Asmiati et al. determined locating chromatic number for amalgamation of star [3] and firecracker graphs [4]. Asmiati et al. successfully found the locating chromatic number for the amalgamation of star [3], firecracker graph [4]. In 2012, Asmiati and Baskoro [5] characterized the loading graph of cycles with three location chromatic numbers, nonhomogeneous caterpillars and firecracker graphs [6], barbell shadow path graphs [7]. In 2017, Asmiati et al. studied the locating chromatic number of n star amalgamations connected by a path [8], and Asmiati et al. for the joint locating chromatic number of several double star graphs [9].

The locating chromatic number for operation graph is also an interesting topic to research. In 2018, Asmiati obtained the locating chromatic number of certain barbell graphs [10]. Next, Irawan et al. for generalized Petersen graph [11] and [12]. Furthermore, Irawan et al. for the origami graph and its barbell [13], [14]. Damayanti et al. for the modified path with cycles [15], Prawinasti et al. for the split cycle graphs [16], and Rahmatalia et al. for split path graphs [17]. The study of locating chromatic number for shadow graphs was carried out by Sudarsana et al. [18].

No research has discussed the locating chromatic number on the Rose graph, as far as the literature search has been carried out. Therefore, the locating chromatic number of the Rose graph and the barbell Rose graphs are discussed in this paper.

2. Basic Properties

Let $G = (V, E)$ be a connected graph and c is a vertex coloring of G such that for any two adjacent vertices u and v in G where $c(u) \neq c(v)$. Let C_p be a color classes, then $\prod = \{C_1, C_2, C_3, \ldots, C_k\}$ is a partition of $V(G)$ that is induced by the coloring of c. The color code of v, c_Π is the k-ordered values $\{d(v, C_1), d(v, C_2), \ldots, d(v, C_k)\}$ with $d(v, C_p) = \min\{d(v, x) \mid x \in C_p\}$ for $1 \leq p \leq k$. If each vertex in Ghas a different color code, then c is called a locating coloring of G. The locating chromatic number of G is denoted by $\chi_L(G)$ is the smallest number k such that G has a locating coloring.

Chartrand et al. [1] gave some characterization for the locating chromatic coloring of a graph in Theorem 2.1.

Theorem 2.1. [1] *Let c be the locating coloring of a connected graph G. Let the set of neighbors of a vertex f in G, denoted by $N(f)$, for all $f \in G$. If f and g are two distinct vertices in G such that $d(g, h) = d(f, h)$ for every $h \in V(G) - \{f, g\}$, then $c(f) \neq c(g)$. In particular, if f and g are not adjacent vertices in G such that $N(f) = N(g)$, then $c(f) \neq c(g)$.*

This theorem showed some restrictions in the construction of vertex coloring of a graph.

Theorem 2.2. [1] *The locating chromatic number of a cycle graph $C_n (n \geq 3)$ is 3 for odd n and 4 for even n.*

Suppose that there is a cycle graph C_n with n vertices, with the vertex set $V(C_n) = \{u_i | 1 \leq i \leq n\}$. The Rose graph is obtained by adding n vertices, where each new vertex is adjacent to two consecutive vertices in the cycle. The Rose graph is a connected graph containing a cycle, with n vertices of degree 2 and n vertices of degree 4, denoted by $M(C_n)$ [19]. Let $\{v_p; p = 1, 2, \ldots, n\} \cup \{w_p; p = 1, 2, \ldots, n\}$ be a set of vertices of $M(C_n)$ and the set of edges is $\{v_p v_{p+1}; p = 1, 2, \ldots, n\} \cup \{v_p w_p; p = 1, 2, \ldots, n\} \cup \{v_{p+1} w_p; p = 1, 2, \ldots, n\}$ with $v_{n+1} = v_1$. The Rose barbell graph is a graph constructed by connecting two Rose graphs $M(C_n)$ and $M'(C_n)$ by an edge $(v_1 v_1')$ as a bridge, denoted by $B_{M(C_n)}$. The following will give a barbell Rose graph $B_{M(C_5)}$ in Figure 1.

3. Research Methods

The methodology used in this study is divided into three steps. Stage 1 is a literature review, namely a literature search on locating chromatic number of a graph. Stage 2 is to find the lower and upper bound of locating chromatic number for Rose graph and its barbell. Furthermore, Stage 3 establishes the theorem gained in Stage 2.

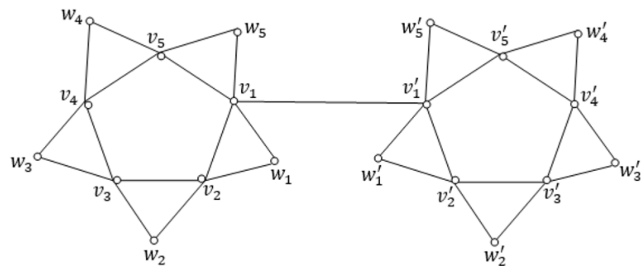

Figure 1. Barbell Rose graph of $M(C_5)$

4. Results and Discussion

To determine the locating chromatic number of a Rose graph $(M(C_n))$ it is essential to analyze how the structural properties of the graph affect vertex colorings that distinguish every pair of vertices by their color code. The following theorem presents the locating chromatic number for Rose graphs of order $n \geq 3$

Theorem 4.1. *If the locating chromatic number of Rose graph $(M(C_n))$ for $n \geq 3$, then*

$$\chi_L(M(C_n)) = \begin{cases} 4; & \text{if } n = \{3, 4\} \\ 5; & \text{if } n \geq 5. \end{cases}$$

Proof. The proof consists of two cases.
Case 1.
Subcase 1.1$(n = 3)$.
Since $M(C_3)$ contains C_3, then by Theorem 2.2 $\chi_L(M(C_3)) \geq 3$. Suppose c is a location coloring of $M(C_3)$ and assume it uses three colors. Without loss of generality, suppose $c(v_1) = c(v_3) = 1$, then $\{c(v_2), c(w_1)\} = \{2, 3\}$. As a result, $c(w_2) = 3$, so that $c_\Pi(w_1) = c_\Pi(w_2)$, a contrary. So, $\chi_L(M(C_3)) \geq 4$.

Let c be a vertex coloring using four colors as follows:

$$c(v_p) = \begin{cases} 1, & p = 1; \\ 2, & p = 2; \\ 3, & p = 3. \end{cases} \qquad c(w_p) = \begin{cases} 1, & p = 2; \\ 2, & p = 3; \\ 4, & p = 1. \end{cases}$$

the color codes of $M(C_3)$:

$$c_\pi(v_p) = \begin{cases} 0, & \text{for } 1^{\text{st}} \text{ value, } p = 1; \\ & \text{for } 2^{\text{nd}} \text{ value, } p = 2; \\ & \text{for } 3^{\text{rd}} \text{ value, } p = 3. \\ 1, & \text{for } 1^{\text{st}} \text{ value, } p = 2, 3; \\ & \text{for } 2^{\text{nd}} \text{ value, } p = 1, 3; \\ & \text{for } 3^{\text{rd}} \text{ value, } p = 1, 2; \\ & \text{for } 4^{\text{th}} \text{ value, } p = 1, 2; \\ 2, & \text{for } 4^{\text{th}} \text{ value, } p = 3. \end{cases} \qquad c_\pi(w_p) = \begin{cases} 0, & \text{for } 1^{\text{st}} \text{ value, } p = 2; \\ & \text{for } 2^{\text{nd}} \text{ value, } p = 3; \\ & \text{for } 4^{\text{th}} \text{ value, } p = 1. \\ 1, & \text{for } 1^{\text{st}} \text{ value, } p = 1, 3; \\ & \text{for } 2^{\text{nd}} \text{ value, } p = 1, 2; \\ & \text{for } 3^{\text{rd}} \text{ value, } p = 2, 3. \\ 2, & \text{for } 3^{\text{rd}} \text{ value, } p = 1; \\ & \text{for } 4^{\text{th}} \text{ value, } p = 2, 3. \end{cases}$$

Since each vertex in $M(C_3)$, has a distinct color code, c is a locating coloring. Consequently, $\chi_L(M(C_3)) \leq 4$. Therefore, $\chi_L(M(C_3)) = 4$.

Subcase 1.2$(n = 4)$.

Since $M(C_4)$ contains C_4, then by Theorem 2.2, $\chi_L(M(C_4)) \geq 4$. Let c be a vertex coloring using four colors as follows.

$$c(v_p) = \begin{cases} 1, & p = 1; \\ 2, & p = 2; \\ 3, & p = 3; \\ 4, & p = 4. \end{cases} \qquad c(w_p) = \begin{cases} 1, & p = 2; \\ 2, & p = 3, 4; \\ 3, & p = 1. \end{cases}$$

the color codes of $M(C_4)$:

$$c_\pi(v_p) = \begin{cases} 0, & \text{for } 1^{\text{st}} \text{ value, } p = 1; \\ & \text{for } 2^{\text{nd}} \text{ value, } p = 2; \\ & \text{for } 3^{\text{rd}} \text{ value, } p = 3; \\ & \text{for } 4^{\text{th}} \text{ value, } p = 4. \\ 1, & \text{for } 1^{\text{st}} \text{ value, } p = 2, 3, 4; \\ & \text{for } 2^{\text{nd}} \text{ value, } p = 1, 3, 4; \\ & \text{for } 3^{\text{rd}} \text{ value, } p = 1, 2, 4; \\ & \text{for } 4^{\text{th}} \text{ value, } p = 1, 3. \\ 2, & \text{for } 4^{\text{th}} \text{ value, } p = 3. \end{cases} \quad c_\pi(w_p) = \begin{cases} 0, & \text{for } 1^{\text{st}} \text{ value, } p = 2; \\ & \text{for } 2^{\text{nd}} \text{ value, } p = 3, 4; \\ & \text{for } 3^{\text{rd}} \text{ value, } p = 1. \\ 1, & \text{for } 1^{\text{st}} \text{ value, } p = 1, 4; \\ & \text{for } 2^{\text{nd}} \text{ value, } p = 1, 2; \\ & \text{for } 3^{\text{rd}} \text{ value, } p = 2, 3. \\ & \text{for } 4^{\text{th}} \text{ value, } p = 3, 4. \\ 2, & \text{for } 1^{\text{nd}} \text{ value, } p = 3; \\ & \text{for } 3^{\text{rd}} \text{ value, } p = 4; \\ & \text{for } 4^{\text{th}} \text{ value, } p = 1, 2. \end{cases}$$

Since each vertex in $M(C_4)$ has a distinct color code, then c is a locating coloring. Consequently,$\chi_L(M(C_4) \leq 4$. Therefore, $\chi_L(M(C_4)) = 4$.

Case 2 $(n \geq 5)$ Consider two subcases.

Subcase 2.1$(n \geq 5)$ **odd**.

Since $M(C_n)$ contains C_n, then by Theorem 2.2 we have $\chi_L(M(C_n)) \geq 3$ for n odd. Suppose c is a locating coloring of $M(C_n)$ and assume it uses three colors. Without loss of generality, suppose $c(v_1) = 1$ and $c(v_{2p}) = 2$ for $p \geq 1$, then $\{c(v_{2p+1}) = 3$ for $p \geq 1$. Consequently, there exists $c(w_q) = c(w_r)$ with $1 \neq q \neq r \neq n$ and $d(w_q, v_a) = d(w_r, v_a)$ which causes $c_\Pi(w_q) = c_\Pi(w_r)$, a contrary. Assume c uses four colors. Without loss of generality, suppose $c(v_1) = 1$, $c(v_{2p}) = 2$ for $p \geq 1$, and $\{c(v_{2p+1}) = 3$ for $p \geq 1$, then $c(w_q) = 4$. Consequently, there exists $c(w_r) = c(w_s)$ with $1 \neq r \neq s \neq n$ and $d(w_r, v_a) = d(w_s, v_a)$ which causes $c_\Pi(w_r) = c_\Pi(w_s)$, a contrary. As result, $\chi_L(M(C_n)) \geq 5$.

Let c be a vertex coloring using five colors as follows:

$$c(v_p) = \begin{cases} 1, & p = 1, \\ 2, & p \text{ even}, \\ 3, & p \text{ odd and } p > 1. \end{cases} \qquad c(w_p) = \begin{cases} 1, & p \in \{1, n\}, \\ 4, & p = 1, \\ 5, & p = n. \end{cases}$$

the color codes of $M(C_n)$ are:

$$c_\Pi(v_p) = \begin{cases} 0, & \text{for } 1^{\text{st}} \text{ value, } p = 1; \\ & \text{for } 2^{\text{nd}} \text{ value, even } p; \\ & \text{for } 3^{\text{rd}} \text{ value, odd } p. \\ 1, & \text{for } 1^{\text{st}} \text{ value, } p \in (1,n]; \\ & \text{for } 2^{\text{nd}} \text{ value, odd } p; \\ & \text{for } 3^{\text{rd}} \text{ value, even } p; \\ & \text{for } 4^{\text{th}} \text{ value, } p = 1,2; \\ & \text{for } 5^{\text{th}} \text{ value, } p = 1. \\ p-1, & \text{for } 4^{\text{th}} \text{ value, } p = \left[1, \frac{n+1}{2}+1\right); \\ p, & \text{for } 5^{\text{th}} \text{ value, } p = \left[1, \frac{n+1}{2}+1\right); \\ n+2-p, & \text{for } 4^{\text{th}} \text{ value, } p = \left(\frac{n+1}{2}+1, n\right]; \\ n+1-p, & \text{for } 5^{\text{th}} \text{ value, } p = \left(\frac{n+1}{2}+1, n\right]. \end{cases} \quad c_\Pi(w_p) = \begin{cases} 0, & \text{for } 1^{\text{st}} \text{ value, } p = (1,n); \\ & \text{for } 4^{\text{th}} \text{ value, } p = 1; \\ & \text{for } 5^{\text{th}} \text{ value, } p = n. \\ 1, & \text{for } 1^{\text{st}} \text{ value, } p = 1,n; \\ & \text{for } 2^{\text{nd}} \text{ value, } p = [1,n); \\ & \text{for } 3^{\text{rd}} \text{ value, } p = (1,n]. \\ 2, & \text{for } 2^{\text{nd}} \text{ value, } p = n; \\ & \text{for } 3^{\text{rd}} \text{ value, } p = 1,n. \\ p, & \text{for } 4^{\text{th}} \text{ value, } p = \left(1, \frac{n+1}{2}\right]; \\ p+1, & \text{for } 5^{\text{th}} \text{ value, } p = \left[1, \frac{n+1}{2}\right]; \\ n+2-p, & \text{for } 4^{\text{th}} \text{ value, } p = \left(\frac{n+1}{2}, n\right]; \\ n+1-p, & \text{for } 5^{\text{th}} \text{ value, } p = \left(\frac{n+1}{2}, n\right). \end{cases}$$

Since each vertex in $M(C_5)$ for odd $n \geq 5$ has a distinct color code, c is a locating coloring using five colors. Consequently,$\chi_L(M(C_n) \leq 5$. Therefore, $\chi_L(M(C_n)) = 5$.

Subcase 2.2$(n \geq 5)$ **even**.

Since $M(C_n)$ contains C_n, then by Theorem 2.2, $\chi_L(M(C_n)) \geq 4$ for n even. Suppose c is a locating coloring of $M(C_n)$ and assume it uses four colors. Without loss of generality, suppose $c(v_1) = 1$ and $c(v_{2p}) = 2$ for $p \geq 1$, and $\{c(v_{2p+1}) = 3$ for $p \geq 1$, then $c(w_q) = 4$. Consequently, there exists $c(w_r) = c(w_s)$ with $1 \neq r \neq s \neq n$ and $d(w_r, v_a) = d(w_s, v_a)$ which causes $c_\Pi(w_r) = c_\Pi(w_s)$, a contrary. As result, $\chi_L(M(C_n)) \geq 5$.

Let c be a vertex coloring using five colors, we obtain:

$$c(v_p) = \begin{cases} 1, & \text{even } p; \\ 2, & \text{odd } p. \end{cases} \qquad c(w_p) = \begin{cases} 3, & p = (1,n); \\ 4, & p = 1; \\ 5, & p = n. \end{cases}$$

the color codes of $M(C_n)$ are:

$$c_\Pi(v_p) = \begin{cases} 0, & \text{for } 1^{\text{st}} \text{ value, odd } p; \\ & \text{for } 2^{\text{nd}} \text{ value, even } p. \\ 1, & \text{for } 1^{\text{st}} \text{ value, even } p; \\ & \text{for } 2^{\text{nd}} \text{ value, odd } p; \\ & \text{for } 3^{\text{rd}} \text{ value, } p = (1,n]; \\ & \text{for } 4^{\text{th}} \text{ value, } p = 1. \\ 2, & \text{for } 3^{\text{rd}} \text{ value, } p = 1. \\ p-1, & \text{for } 4^{\text{th}} \text{ value, } p = \left(1, \frac{n}{2}+1\right]; \\ p, & \text{for } 5^{\text{th}} \text{ value, } p = \left[1, \frac{n}{2}\right]; \\ n-p+2, & \text{for } 4^{\text{th}} \text{ value, } p = \left(\frac{n}{2}+1, n\right]; \\ n-p+1, & \text{for } 5^{\text{th}} \text{ value, } p = \left(\frac{n}{2}, n\right]. \end{cases} \quad c_\Pi(w_p) = \begin{cases} 0, & \text{for } 3^{\text{rd}} \text{ value, } p = (1,n); \\ & \text{for } 4^{\text{th}} \text{ value, } p = 1; \\ & \text{for } 5^{\text{th}} \text{ value, } p = n. \\ 1, & \text{for } 1^{\text{st}} \text{ value, } p = [1,n]; \\ & \text{for } 2^{\text{nd}} \text{ value, } p = [1,n]; \\ 2, & \text{for } 3^{\text{rd}} \text{ value, } p = 1,n. \\ p, & \text{for } 4^{\text{th}} \text{ value, } p = \left(1, \frac{n}{2}+1\right]; \\ p+1, & \text{for } 5^{\text{th}} \text{ value, } p = \left[1, \frac{n}{2}\right]; \\ n-p+2, & \text{for } 4^{\text{th}} \text{ value, } p = \left(\frac{n}{2}+1, n\right]; \\ n-p+1, & \text{for } 5^{\text{th}} \text{ value, } p = \left(\frac{n}{2}, n\right). \end{cases}$$

Since each vertex in $M(C_n)$ for even $n \geq 5$ has a distinct color code, c is a locating coloring using 5 colors. Consequently, $\chi_L(M(C_n) \leq 5$. Therefore, $\chi_L(M(C_n)) = 5$. □

Figure 2 is an example of a minimum locating coloring of the Rose graph $M(C_5)$ with the locating chromatic number 5.

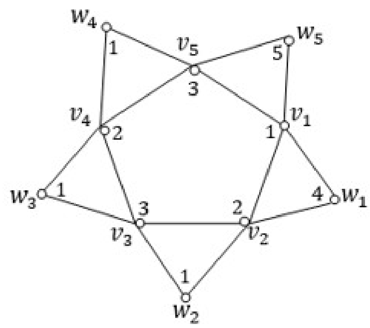

Figure 2. A minimum locating coloring of $M(C_5)$

After determining the locating chromatic number of the Rose graph $M(C_n)$, we now turn our attention to a more complex structure, namely the barbell Rose graph $B_{M(C_n)}$, which consists of two Rose graphs connected by a bridge. The following theorem establishes the locating chromatic number for $n \geq 3$.

Theorem 4.2. *The locating chromatic number of barbell Rose graph $B_{M(C_n)}$ for $n \geq 3$,*

$$\chi_L\left(B_{M(C_n)}\right) = \begin{cases} 4; & \text{if } n = 3 \\ 5; & \text{if } n \geq 4. \end{cases}$$

Proof. The proof consists of two cases.
Case 1(n=3) Since the barbell Rose graph $B_{M(C_3)}$ contains the Rose graph $M(C_3)$, then by Theorem 4.1 we have $\chi_L\left(B_{M(C_3)}\right) \geq 4$.

Let c be a vertex coloring using four colors as follows

$$c(v_p) = \begin{cases} 1, & p = 1; \\ 2, & p = 2; \\ 3, & p = 3. \end{cases} \quad c(w_p) = \begin{cases} 1, & p = 2; \\ 2, & p = 3; \\ 4, & p = 1. \end{cases} \quad c(v'_p) = \begin{cases} 2, & p = 1; \\ 3, & p = 2; \\ 4, & p = 3; \end{cases} \quad c(w'_p) = \begin{cases} 1, & p = 2; \\ 3, & p = 1; \\ 4, & p = 3; \end{cases}$$

the color codes of $B_{M(C_3)}$ are:

$$c_\pi(v_p) = \begin{cases} 0, & \text{for } 1^{\text{st}} \text{ value, } p = 1; \\ & \text{for } 2^{\text{nd}} \text{ value, } p = 2; \\ & \text{for } 3^{\text{rd}} \text{ value, } p = 3. \\ 1, & \text{for } 1^{\text{st}} \text{ value, } p = 2,3; \\ & \text{for } 2^{\text{nd}} \text{ value, } p = 1,3; \\ & \text{for } 3^{\text{rd}} \text{ value, } p = 1,2; \\ & \text{for } 4^{\text{th}} \text{ value, } p = 1,2. \\ 2, & \text{for } 4^{\text{th}} \text{ value, } p = 3. \end{cases} \qquad c_\pi(w_p) = \begin{cases} 0, & \text{for } 1^{\text{st}} \text{ value, } p = 2; \\ & \text{for } 2^{\text{nd}} \text{ value, } p = 3; \\ & \text{for } 4^{\text{th}} \text{ value, } p = 1. \\ 1, & \text{for } 1^{\text{st}} \text{ value, } p = 1,3; \\ & \text{for } 2^{\text{nd}} \text{ value, } p = 1,2; \\ & \text{for } 3^{\text{rd}} \text{ value, } p = 2,3, \\ 2, & \text{for } 3^{\text{rd}} \text{ value, } p = 1; \\ & \text{for } 4^{\text{th}} \text{ value, } p = 2,3. \end{cases}$$

$$c_\pi(v'_p) = \begin{cases} 0, & \text{for } 2^{\text{nd}} \text{ value, } p = 1; \\ & \text{for } 3^{\text{rd}} \text{ value, } p = 2; \\ & \text{for } 4^{\text{th}} \text{ value, } p = 3. \\ 1, & \text{for } 1^{\text{st}} \text{ value, } p = 2,3; \\ & \text{for } 2^{\text{nd}} \text{ value, } p = 2,3; \\ & \text{for } 3^{\text{rd}} \text{ value, } p = 1,3; \\ & \text{for } 4^{\text{th}} \text{ value, } p = 1,2. \\ 2, & \text{for } 1^{\text{st}} \text{ value, } p = 1. \end{cases} \qquad c_\pi(w'_p) = \begin{cases} 0, & \text{for } 1^{\text{st}} \text{ value, } p = 2; \\ & \text{for } 3^{\text{rd}} \text{ value, } p = 1; \\ & \text{for } 4^{\text{th}} \text{ value, } p = 3. \\ 1, & \text{for } 2^{\text{nd}} \text{ value, } p = 1,3; \\ & \text{for } 3^{\text{3d}} \text{ value, } p = 2,3; \\ & \text{for } 4^{\text{th}} \text{ value, } p = 1,2; \\ 2, & \text{for } 1^{\text{st}} \text{ value, } p = 1,3. \\ & \text{for } 2^{\text{nd}} \text{ value, } p = 2. \end{cases}$$

Since each vertex in $B_{M(C_3)}$ has a distinct color code, c is a locating coloring. Consequently,$\chi_L\left(B_{M(C_3)}\right) \leq 4$. Therefore, $\chi_L\left(B_{M(C_3)}\right) = 4$.

Case 2$(n \geq 4)$ Consider three subcases.

Subcase 2.1$(n = 4)$.

First, we determine the lower bound of $\chi_L\left(B_{M(C_4)}\right)$. Since the barbell Rose graph $B_{M(C_4)}$ contains the Rose graph $M(C_4)$, then by Theorem 4.1 we have $\chi_L\left(B_{M(C_3)}\right) \geq 4$. Without loss of generality, suppose $c(v_1) = 1$ and $c(v_2) = c(v_4) = c(v'_1) = 2$ and $c(w_1) = c(w_4) = c(v'_2) = c(v'_4) = 3$, then $\{c(w'_1), c(w'_4)\} = \{1, 4\} \geq 5$. Consequently $c_\Pi(v_1) = c_\Pi(w'_1)$, a contrary. As a result, $\chi_L\left(B_{M(C_3)}\right) \geq 5$.

Let c be a vertex coloring using five colors as follows:

$$c(v_p) = \begin{cases} 1, & p = 1; \\ 2, & p = 2; \\ 3, & p = 3; \\ 3, & p = 4. \end{cases} \qquad c(w_p) = \begin{cases} 1, & p = 2; \\ 2, & p = 3,4; \\ 4, & p = 1. \end{cases} \qquad c(v'_p) = \begin{cases} 2, & p = 2; \\ 3, & p = 3; \\ 4, & p = 4; \\ 5, & p = 1. \end{cases} \qquad c(w'_p) = \begin{cases} 1, & p = 2,3; \\ 3, & p = 4; \\ 4, & p = 1. \end{cases}$$

the color codes of $B_{M(C_4)}$ are:

$$c_\pi(v_p) = \begin{cases} 0, & \text{for } 1^{\text{st}} \text{ value, } p = 1; \\ & \text{for } 2^{\text{nd}} \text{ value, } p = 2; \\ & \text{for } 3^{\text{rd}} \text{ value, } p = 3; \\ & \text{for } 4^{\text{th}} \text{ value, } p = 4. \\ 1, & \text{for } 1^{\text{st}} \text{ value, } p = 2,3,4; \\ & \text{for } 2^{\text{nd}} \text{ value, } p = 1,3,4; \\ & \text{for } 3^{\text{rd}} \text{ value, } p = 1,4,2; \\ & \text{for } 4^{\text{th}} \text{ value, } p = 1,3; \\ & \text{for } 5^{\text{th}} \text{ value, } p = 1. \\ 2, & \text{for } 4^{\text{th}} \text{ value, } p = 2; \\ & \text{for } 5^{\text{th}} \text{ value, } p = 2,4. \\ 3, & \text{for } 5^{\text{th}} \text{ value, } p = 3. \end{cases} \qquad c_\pi(w_p) = \begin{cases} 0, & \text{for } 1^{\text{st}} \text{ value, } p = 2,4; \\ & \text{for } 2^{\text{nd}} \text{ value, } p = 3; \\ & \text{for } 4^{\text{th}} \text{ value, } p = 1. \\ 1, & \text{for } 1^{\text{st}} \text{ value, } p = 1,4; \\ & \text{for } 2^{\text{nd}} \text{ value, } p = 1,2; \\ & \text{for } 3^{\text{rd}} \text{ value, } p = 2,3; \\ & \text{for } 4^{\text{th}} \text{ value, } p = 3,4. \\ 2, & \text{for } 1^{\text{st}} \text{ value, } p = 3; \\ & \text{for } 3^{\text{rd}} \text{ value, } p = 4; \\ & \text{for } 4^{\text{th}} \text{ value, } p = 1,2; \\ & \text{for } 5^{\text{th}} \text{ value, } p = 1,4. \\ 3 & \text{for } 5^{\text{th}} \text{ value, } p = 2,3. \end{cases}$$

$$c_\pi(v'_p) = \begin{cases} 0, & \text{for } 2^{\text{nd}} \text{ value}, \ p = 2; \\ & \text{for } 3^{\text{rd}} \text{ value}, \ p = 3; \\ & \text{for } 4^{\text{th}} \text{ value}, \ p = 4; \\ & \text{for } 5^{\text{th}} \text{ value}, \ p = 1. \\ 1, & \text{for } 1^{\text{st}} \text{ value}, \ p = 1,2,3,4; \\ & \text{for } 2^{\text{nd}} \text{ value}, \ p = 1,3,4; \\ & \text{for } 3^{\text{rd}} \text{ value}, \ p = 1,2,4; \\ & \text{for } 4^{\text{th}} \text{ value}, \ p = 1,3. \\ & \text{for } 5^{\text{th}} \text{ value}, \ p = 2,4. \\ 2, & \text{for } 4^{\text{th}} \text{ value}, \ p = 2; \\ & \text{for } 5^{\text{th}} \text{ value}, \ p = 3. \end{cases} \quad c_\pi(w'_p) = \begin{cases} 0, & \text{for } 1^{\text{st}} \text{ value}, \ p = 2,3; \\ & \text{for } 2^{\text{rd}} \text{ value}, \ p = 4; \\ & \text{for } 3^{\text{th}} \text{ value}, \ p = 1. \\ 1, & \text{for } 2^{\text{nd}} \text{ value}, \ p = 1,2; \\ & \text{for } 3^{\text{3d}} \text{ value}, \ p = 2,3; \\ & \text{for } 4^{\text{th}} \text{ value}, \ p = 3,4; \\ & \text{for } 5^{\text{th}} \text{ value}, \ p = 1,5. \\ 2, & \text{for } 1^{\text{st}} \text{ value}, \ p = 1,4. \\ & \text{for } 2^{\text{nd}} \text{ value}, \ p = 3. \\ & \text{for } 3^{\text{3d}} \text{ value}, \ p = 4; \\ & \text{for } 4^{\text{th}} \text{ value}, \ p = 1,2; \\ & \text{for } 5^{\text{th}} \text{ value}, \ p = 2,3. \end{cases}$$

Since each vertex in $B_{M(C_4)}$ has a distinct color code, c is a locating coloring. Consequently,$\chi_L\left(B_{M(C_4)}\right) \leq 5$. Therefore, $\chi_L\left(B_{M(C_3)}\right) = 5$.

Subcase 2.2 for odd $(n > 4)$.
Since the barbell Rose graph $B_{M(C_n)}$ contains the Rose graph $M(C_n)$,then by Theorem 4.1 we have $\chi_L\left(B_{M(C_n)}\right) \geq 5$.

Let c be a vertex coloring using five colors as follows.

$$c(v_p) = \begin{cases} 1, & p = 1; \\ 2, & \text{even } p; \\ 3, & \text{odd } p > 1. \end{cases} \qquad c(w_p) = \begin{cases} 1, & p = (1,n); \\ 2, & p = n; \\ 4, & p = 1. \end{cases}$$

$$c(v'_p) = \begin{cases} 1, & p = n; \\ 2, & \text{odd } p > 2, p \neq n; \\ 3, & \text{even } p. \end{cases} \qquad c(w'_p) = \begin{cases} 1, & p = [1, n-2]; \\ 2, & p = n-1; \\ 5, & p = n. \end{cases}$$

the color codes of $B_{M(C_n)}$ are:

$$c_\Pi(v_p) = \begin{cases} 0, & \text{for } 1^{\text{st}} \text{ value, } p = 1; \\ & \text{for } 2^{\text{nd}} \text{ value, even } p. \\ & \text{for } 3^{\text{rd}} \text{ value, odd } p > 1. \\ 1, & \text{for } 1^{\text{st}} \text{ value, } p = (1, n]; \\ & \text{for } 2^{\text{nd}} \text{ value, odd } p; \\ & \text{for } 3^{\text{rd}} \text{ value, even } p = 1; \\ & \text{for } 4^{\text{th}} \text{ value, } p = 1. \\ p-1, & \text{for } 4^{\text{th}} \text{ value, } p = \left(1, \frac{n+1}{2}+1\right]. \\ p+1, & \text{for } 5^{\text{th}} \text{ value, } p = \left[\frac{n+1}{2}+1, n\right]. \\ n-p+2, & \text{for } 4^{\text{th}} \text{ value, } p = \left(\frac{n+1}{2}+1, n\right]. \\ n-p+3, & \text{for } 5^{\text{th}} \text{ value, } p = \left(\frac{n+1}{2}, n\right]. \end{cases} \quad c_\Pi(w_p) = \begin{cases} 0, & \text{for } 1^{\text{st}} \text{ value, } p = (1, n); \\ & \text{for } 2^{\text{nd}} \text{ value, } p = n. \\ & \text{for } 4^{\text{th}} \text{ value, } p = 1. \\ 1, & \text{for } 1^{\text{st}} \text{ value, } p = 1, n; \\ & \text{for } 2^{\text{nd}} \text{ value, } p = [1, n); \\ & \text{for } 3^{\text{rd}} \text{ value, } p = (1, n]. \\ 2, & \text{for } 3^{\text{rd}} \text{ value, } p = 1. \\ p, & \text{for } 4^{\text{th}} \text{ value, } p = \left(1, \frac{n+1}{2}\right]. \\ p+2, & \text{for } 5^{\text{th}} \text{ value, } p = \left[1, \frac{n+1}{2}\right]. \\ n-p+2, & \text{for } 4^{\text{th}} \text{ value, } p = \left(\frac{n+1}{2}, n\right]. \\ n-p+3, & \text{for } 5^{\text{th}} \text{ value, } p = \left(\frac{n+1}{2}, n\right]. \end{cases}$$

$$c_\Pi(v'_p) = \begin{cases} 0, & \text{for } 1^{\text{st}} \text{ value, } p = n; \\ & \text{for } 2^{\text{nd}} \text{ value, odd } p, p \neq n. \\ & \text{for } 3^{\text{rd}} \text{ value, even } p. \\ 1, & \text{for } 1^{\text{st}} \text{ value, } p = [1, n); \\ & \text{for } 2^{\text{nd}} \text{ value, odd } p = n; \\ & \text{for } 3^{\text{rd}} \text{ value, even } p. \\ p+1 & \text{for } 4^{\text{th}} \text{ value, } p = \left[1, \frac{n+1}{2}\right]. \\ p, & \text{for } 5^{\text{th}} \text{ value, } p = \left[1, \frac{n+1}{2}\right]. \\ n-p+3, & \text{for } 4^{\text{th}} \text{ value, } p = \left(\frac{n+1}{2}, n\right]. \\ n-p+1, & \text{for } 5^{\text{th}} \text{ value, } p = \left(\frac{n+1}{2}, n\right]. \end{cases} \quad c_\Pi(w'_p) = \begin{cases} 0, & \text{for } 1^{\text{st}} \text{ value, } p = [1, n-2]; \\ & \text{for } 2^{\text{nd}} \text{ value, } p = n-1; \\ & \text{for } 5^{\text{th}} \text{ value, } p = n. \\ 1, & \text{for } 1^{\text{st}} \text{ value, } p = n-1, n; \\ & \text{for } 2^{\text{nd}} \text{ value, } p = [1, n] - \{n-1\}; \\ & \text{for } 3^{\text{rd}} \text{ value, } p = [1, n). \\ 2, & \text{for } 3^{\text{rd}} \text{ value, } p = n. \\ p+2, & \text{for } 4^{\text{th}} \text{ value, } p = \left[1, \frac{n+1}{2}\right]. \\ p+1, & \text{for } 5^{\text{th}} \text{ value, } p = \left[1, \frac{n-1}{2}\right]. \\ n-p+3, & \text{for } 4^{\text{th}} \text{ value, } p = \left(\frac{n+1}{2}, n\right]. \\ n-p+1, & \text{for } 5^{\text{th}} \text{ value, } p = \left(\frac{n-1}{2}, n\right]. \end{cases}$$

Since each vertex in $B_{M(C_n)}$ has a distinct color code, c is a locating coloring. Consequently,$\chi_L\left(B_{M(C_n)}\right) \leq 5$. Therefore, $\chi_L\left(B_{M(C_n)}\right) = 5$.

Subcase 2.3 for even $(n > 4)$.
Since the barbell Rose graph $B_{M(C_n)}$ contains the Rose graph $M(C_n)$, then by Theorem 4.1 we have $\chi_L\left(B_{M(C_n)}\right) \geq 5$.

Let c be a vertex coloring using five colors as follows.

$$c(v_p) = \begin{cases} 1, & \text{odd } p; \\ 2, & \text{even } p. \end{cases} \quad c(w_p) = \begin{cases} 3, & p = [1, n); \\ 4, & p = n; \end{cases} \quad c(v'_p) = \begin{cases} 1, & \text{even } p; \\ 2, & \text{odd } p. \end{cases} \quad c(w'_p) = \begin{cases} 3, & p = [1, n); \\ 5, & p = n. \end{cases}$$

the color codes of $B_{M(C_n)}$ are:

$$c_\Pi(v_p)=\begin{cases}0, & \text{for } 1^{\text{st}} \text{ value, odd } p;\\ & \text{for } 2^{\text{nd}} \text{ value, even } p.\\ 1, & \text{for } 1^{\text{st}} \text{ value, even } p;\\ & \text{for } 2^{\text{nd}} \text{ value, odd } p;\\ & \text{for } 3^{\text{rd}} \text{ value, even } p=[1,n];\\ p, & \text{for } 4^{\text{th}} \text{ value, } p=p=\left[1,\frac{n}{2}\right].\\ p+1, & \text{for } 5^{\text{th}} \text{ value, } p=\left[1,\frac{n}{2}+1\right].\\ n-p+1, & \text{for } 4^{\text{th}} \text{ value, } p=\left(\frac{n}{2},n\right].\\ n-p+3, & \text{for } 5^{\text{th}} \text{ value, } p=\left(\frac{n}{2}+1,n\right].\end{cases}
\qquad
c_\Pi(w_p)=\begin{cases}0, & \text{for } 3^{\text{rd}} \text{ value, } p=[1,n);\\ & \text{for } 4^{\text{th}} \text{ value, } p=n.\\ 1, & \text{for } 1^{\text{st}} \text{ value, } p=[1,n];\\ & \text{for } 2^{\text{nd}} \text{ value, } p=[1,n].\\ 2, & \text{for } 3^{\text{rd}} \text{ value, } p=n.\\ p+1, & \text{for } 4^{\text{th}} \text{ value, } p=\left[1,\frac{n}{2}\right].\\ p+2, & \text{for } 5^{\text{th}} \text{ value, } p=\left[1,\frac{n}{2}\right].\\ n-p+2, & \text{for } 4^{\text{th}} \text{ value, } p=\left(\frac{n}{2},n\right).\\ n-p+3, & \text{for } 5^{\text{th}} \text{ value, } p=\left(\frac{n}{2},n\right].\end{cases}$$

$$c_\Pi(v'_p)=\begin{cases}0, & \text{for } 1^{\text{st}} \text{ value, even } p;\\ & \text{for } 2^{\text{nd}} \text{ value, odd } p.\\ 1, & \text{for } 1^{\text{st}} \text{ value, odd } p;\\ & \text{for } 2^{\text{nd}} \text{ value, even } p;\\ & \text{for } 3^{\text{rd}} \text{ value, } p=[1,n].\\ p+1 & \text{for } 4^{\text{th}} \text{ value, } p=\left[1,\frac{n}{2}+1\right].\\ p, & \text{for } 5^{\text{th}} \text{ value, } p=\left[1,\frac{n}{2}\right].\\ n-p+3, & \text{for } 4^{\text{th}} \text{ value, } p=\left(\frac{n}{2}+1,n\right].\\ n-p+1, & \text{for } 5^{\text{th}} \text{ value, } p=\left(\frac{n}{2},n\right].\end{cases}
\qquad
c_\Pi(w'_p)=\begin{cases}0, & \text{for } 3^{\text{st}} \text{ value, } p=[1,n);\\ & \text{for } 5^{\text{th}} \text{ value, } p=n.\\ 1, & \text{for } 1^{\text{st}} \text{ value, } p=[1,n];\\ & \text{for } 2^{\text{nd}} \text{ value, } p=[1,n].\\ 2, & \text{for } 3^{\text{rd}} \text{ value, } p=n.\\ p+2, & \text{for } 4^{\text{th}} \text{ value, } p=\left[1,\frac{n}{2}\right].\\ p+1, & \text{for } 5^{\text{th}} \text{ value, } p=\left[1,\frac{n}{2}\right].\\ n-p+3, & \text{for } 4^{\text{th}} \text{ value, } p=\left(\frac{n}{2},n\right].\\ n-p+1, & \text{for } 5^{\text{th}} \text{ value, } p=\left(\frac{n}{2},n\right).\end{cases}$$

Since each vertex in $B_{M(C_n)}$ has a distinct color code, c is a locating coloring. Consequently,$\chi_L\left(B_{M(C_n)}\right)\leq 5$. Therefore, $\chi_L\left(B_{M(C_n)}\right)=5$.

□

Figure 3 is an example of a minimum locating coloring of the barbell Rose graph $B_{M(C_5)}$.

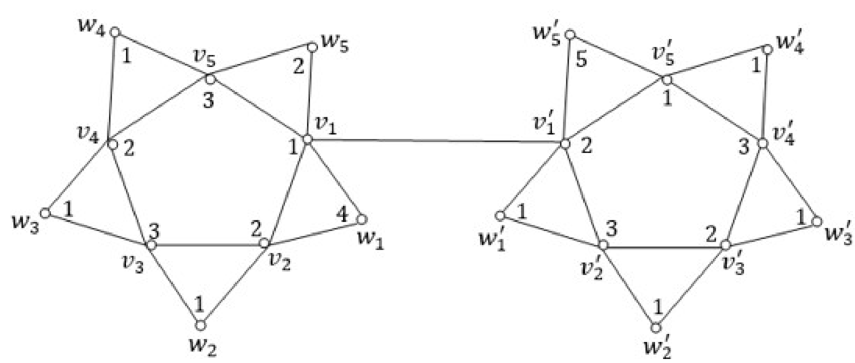

Figure 3. A minimum locating coloring of $B_{M(C_5)}$

5. Conclusions

Determining the locating chromatic number of a Rose graph and barbell Rose graph depends on the number of vertices in its cycle. Based on the results of the discussion, it was found that the locating chromatic number of Rose graph for $n \in \{3,4\}$ is 4 and $n \geq 5$ is 5. The same thing applied to the locating chromatic number of barbell Rose graph, namely $\chi_L\left(B_{M(C_3)}\right) = 4$ and 5 for $n \geq 4$.

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