

# On the Iterated Cevian Triangle in Finite Euclidean Space

Arfah<sup>1\*</sup>

<sup>1</sup>Department of Mathematics, State University of Makassar, Indonesia

\*Corresponding author: [arfah@unm.ac.id](mailto:arfah@unm.ac.id)

## A B S T R A C T

This paper investigates the iterative process of constructing Cevian triangles in a finite Euclidean plane. First, we prove that starting from an initial triangle, the process of iteratively taking Cevian triangles converges to a unique fixed point. Second, we show this convergence process is surjective onto the interior of the triangle; that is, for any target point in the interior, one can find an initial point whose sequence of iterated Cevian triangles converges to that target. Finally, we examine the limiting configuration of an infinite iteration and characterize a novel property intrinsic to the discrete nature of the finite geometric space, setting it apart from the classical real Euclidean case.

**Keywords:** Cevian triangle, iterations, Euclidean space.

Received : 11-10-2025;  
 Revised : 25-11-2025;  
 Accepted : 09-12-2025;  
 Published : 02-03-2026;

DOI: <https://doi.org/10.29303/emj.v9i1.336>



This work is licensed under a [CC BY-NC-SA 4.0 International](https://creativecommons.org/licenses/by-nc-sa/4.0/) license

## 1. Introduction

The study of triangle geometry, with its rich history dating back to the ancient Greeks, continues to be a fertile ground for discovery in modern mathematics. Among its most elegant constructs is the cevian triangle—the triangle formed by the feet of the cevians drawn from a point to the sides of a given triangle. A line segment joining a vertex of a triangle to any given point on the opposite side is called a Cevian. The triangle formed by connecting the points on the opposite side of the triangle vertex in the Cevian lines is called Cevian triangle.

Seebach [1] in 1987 and Muaffaq [2] in 2006 proved in different way that if  $\Delta(A, B, C)$  and  $\Delta(P, Q, R)$  are arbitrary triangles, then there exists unique triangle  $\Delta(A', B', C')$  on the sides of the triangle  $\Delta(A, B, C)$  such that the triangle  $\Delta(A', B', C')$  is similar to the triangle  $\Delta(P, Q, R)$ . In this case,  $\Delta(A', B', C')$  is said to be the Cevian triangle of  $\Delta(A, B, C)$ . Next, in 2015, Hvala [3] considered the Seebach theorem that there exist six points inside a triangle with Cevian triangles similar to the reference triangle. He showed an access to these five points using an additional tool: a possibility to draw a conic through five given points. Furthermore, Aliyef [4] proved generalization of Schlämilch's

and Zetel's theorems about concurrent lines in a cevian triangle. This generalization is obtained as a corollary of sharp geometric inequality about the ratio of triangular areas which is proved using discrete variant of Hölder's inequality.

Classical results, such as Ceva's Theorem, provide a fundamental understanding of concurrency, but the dynamical behavior arising from the iteration of this construction has garnered significant interest in contemporary research. Carrol et.al. [5], for example, they investigated a series of iterated Routh's triangles, and showed that for a random sequence of iterations, the expected value of the limiting point is the centroid of the original triangle. Another work is by Brunck [6]. He consider a series of geodesic triangle on a surface of constant curvature by subdivide it recursively into four triangles by joining the midpoints of its edges. He showed the existence of a uniform  $\delta$  such that, at any step of the subdivision, all the triangle angles lie in the interval  $(\delta, \pi - \delta)$ . Furthermore, he exhibited stabilising behaviours for both angles and lengths as this subdivision progresses.

Inspired from this works, this paper aims to establish a comprehensive theory for iterated cevian triangles within the discrete setting of finite Euclidean spaces. Our primary objectives are threefold: first, to demonstrate that for any initial point, the sequence of iterated cevian triangles converges to a unique interior point under a suitably defined notion of convergence; second, to prove the surjectivity of this process by showing that every point in the triangle's interior is attainable as the limit of such a sequence; and finally, to uncover and characterize a distinctive combinatorial property inherent to the infinite iterated cevian triangle, a feature emergent solely from the underlying finite Euclidean space.

## 2. Preliminaries

In this section, we provide the fundamental theories that will be used to derive our main results.

**Definition 2.1.** [7] A set  $C$  is convex if any geodesic segment joining two points in  $C$  is in  $C$ , i.e., for any  $x, y \in C$ , and any  $\alpha \in [0, 1]$ , we have

$$\alpha x \oplus (1 - \alpha)y \in C.$$

We call  $\bar{x}$  as a convex combination of points  $x_1, x_2, \dots, x_n \in C$  if  $\bar{x}$  can be represented as  $\bar{x} = \alpha_1 x_1 \oplus \alpha_2 x_2 \oplus \dots \oplus \alpha_n x_n$  where  $\alpha_1 + \alpha_2 + \dots + \alpha_n = 1$  and  $\alpha_i \geq 0$  for all  $i = 1, 2, \dots, n$ . Note that an affine set is defined as set containing the entire line through any two distinct points in it. It follows that every affine set is a convex set. As with affine sets, it can be seen that a set is convex if and only if it contains every convex combination of its points.

The convex hull of a set  $C$ , denoted by  $\mathbf{co}C$ , is the set of all convex combinations of points in  $C$ , i.e.,

$$\mathbf{co}C = \left\{ \sum_{i=1}^n \alpha_i x_i : x_i \in C, \sum_{i=1}^n \alpha_i = 1, i = 1, 2, \dots, n \right\}.$$

Let  $\mathbb{E}^m$  denote the  $m$  dimensional Euclidean space. By this definition, we can see that the convex hull formed by points  $A, B$ , and  $C$  is the region restricted by triangle  $\Delta(A, B, C)$  in  $\mathbb{E}^m$ , i.e.,

$$\mathbf{co}\Delta = \{P \in \mathbb{E}^m : P = \alpha A + \beta B + \gamma C; \alpha, \beta, \gamma \in [0, 1]; \alpha + \beta + \gamma = 1\}.$$

**Definition 2.2.** [8] A point  $x$  is an interior point of a set  $S \subset \mathbb{E}^m$  if  $S$  contains some open ball of positive radius centered at  $x$ , that is, there exists a  $\delta > 0$  such that  $B(x, \delta) \subset S$ . The set of all interior points of a set  $S$  is denoted by  $\text{Int}(S)$ .

**Definition 2.3.** [8] A subset  $K$  of  $\mathbb{E}^n$  is compact if every infinite subset of  $K$  has an accumulation point that belongs to  $K$ .

**Theorem 2.4.** [8] A subset  $K$  of  $\mathbb{E}^m$  is compact if and only if

- (i) Every infinite subset of  $K$  has at least one accumulation point  $p$ , and
- (ii) All accumulation points of  $K$  belong to  $K$ .

**Theorem 2.5.** [8] A compact subset  $K$  of  $\mathbb{E}^m$  must be closed and bounded.

**Theorem 2.6.** [8] If  $K_1, K_2, K_3, \dots$  is a family of nonempty compact subsets of  $\mathbb{E}^m$  such that

$$K_1 \supseteq K_2 \supseteq K_3 \supseteq \dots,$$

then

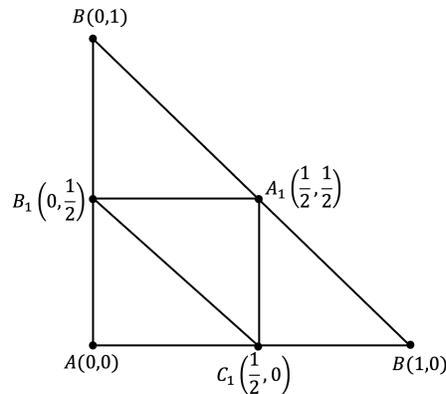
$$\bigcap_{i=1}^{\infty} K_i \neq \emptyset.$$

Next, we provide the following definition of the iterated Cevian triangle as follows.

**Definition 2.7.** The iterated Cevian triangle in  $\mathbb{E}^m$  is a sequence of  $\Delta_0, \Delta_1, \Delta_2, \dots$  refined from triangulations:

- i  $\Delta_0 = \Delta$ ;
- ii  $\Delta_{n+1}$  is obtained by taking any point from each side of  $\Delta_n$  and pairwise connecting these three points.

**Example 2.8.** Let  $A(0,0)$ ,  $B(1,0)$ , and  $C(0,1)$  be the vertices of the triangle  $\Delta(A, B, C)$  in  $\mathbb{E}^2$ . Denote  $A_1, B_1$ , and  $C_1$  the points in the geodesic segments  $[B, C]$ ,  $[A, C]$ , and  $[A, B]$ , respectively. Then we can take  $\alpha, \beta, \gamma \in (0, 1)$  such that  $A_1 = \alpha B \oplus (1 - \alpha)C$ ,  $B_1 = \beta A \oplus (1 - \beta)C$ , and  $C_1 = \gamma A \oplus (1 - \gamma)B$ . Take  $\alpha = \beta = \gamma = 0.5$ , gives us  $A_1 = \left(\frac{1}{2}, \frac{1}{2}\right)$ ,  $B_1 = \left(0, \frac{1}{2}\right)$ , and  $C_1 = \left(\frac{1}{2}, 0\right)$ . Then the triangle  $\Delta_1$  with vertices of  $A_1, B_1$ , and  $C_1$  is the Cevian triangle derived from  $\Delta(A, B, C)$ . Note that  $\Delta_1$  is also called median triangle.



**Figure 1.** Cevian triangle  $\Delta(A_1, B_1, C_1)$ .

### 3. Main Results

In this section, we begin this section with the following lemma before we show the main theorems.

**Lemma 3.1.** Let  $r : [0, l] \rightarrow \mathbb{E}^m$  be a unit geodesic joining points  $P = r(0)$  and  $Q = r(l)$  on a triangle  $\Delta$ . Then the restriction of  $r$  to the open interval  $(0, l)$  lies in the interior of  $\Delta$ .

*Proof.* Let  $\Delta(A, B, C)$  be a triangle in  $\mathbb{E}^m$ . Denote  $[A, B]$  and  $[C, D]$  geodesic segments connecting points  $A$  and  $B$  and points  $C$  and  $D$ , respectively. Take arbitrary two points on  $\Delta(A, B, C)$ , say  $P \in [A, B]$  and  $Q \in [B, C]$ . It follows that  $P = \alpha A \oplus (1 - \alpha)B$  and  $Q = \beta B \oplus (1 - \beta)C$  for  $\alpha, \beta \in [0, 1]$ . Suppose that  $r : [0, 1] \rightarrow [P, Q]$  with  $P \neq Q$ . Let  $\gamma \in (0, 1)$  such that

$$R = \gamma P \oplus (1 - \gamma)Q.$$

As a consequence, we have

$$\begin{aligned} R &= \gamma(\alpha A \oplus (1 - \alpha)B) + (1 - \gamma)(\beta B \oplus (1 - \beta)C) \\ &= \alpha\gamma A + (\gamma(1 - \alpha) + (1 - \gamma)\beta) B + (1 - \gamma)(1 - \beta)C \\ &= k_1 A + k_2 B + k_3 C \end{aligned}$$

where  $k_1 = \alpha\gamma$ ,  $k_2 = \gamma(1 - \alpha) + (1 - \gamma)\beta$ , and  $k_3 = (1 - \gamma)(1 - \beta)$ .

It can be seen that  $k_1 + k_2 + k_3 = 1$ . By definition of  $\mathbf{co}\Delta$  and  $\gamma \in (0, 1)$ ,  $R$  lies in the interior of the triangle  $\Delta(A, B, C)$ .  $\square$

From this lemma, we obtain the following corollary.

**Corollary 3.2.** Let  $\Delta_n$  be the  $n$ -th iterated triangle from  $\Delta(A, B, C)$ . Suppose that  $\Delta_{n+1}$  is the innermost triangle of  $\Delta_n$  by subdivision, i.e, the limit shape of the Cevian nest after infinitely many steps, form two disjoint closed sets. Then the vertices of  $\Delta_n$  and the innermost triangle  $\Delta_{n+1}$  are disjoint.

Next, we show the convergence of the iterated Cevian triangle.

**Theorem 3.3.** Let  $\Delta_n$  be the iterated Cevian triangle in  $\mathbb{E}^m$ . For any  $\varepsilon > 0$ , there exists  $N$  such that all the edge length of  $\Delta_n$  are smaller than  $\varepsilon$  for all  $n > N$ .

*Proof.* Let  $\{\Delta_n\}_{n \in \mathbb{N}}$  be a sequence of nested triangle obtained by iterative process and by Theorem 2.6, we have that  $\bigcap_{i=1}^{\infty} \Delta_i \neq \emptyset$ . We define the sequences  $(A_n)$ ,  $(B_n)$ , and  $(C_n)$ , the vertices of the triangle  $\Delta_n$  incident to the angles  $\alpha_n$ ,  $\beta_n$ , and  $\gamma_n$ , respectively. Note that  $\Delta_n$  is compact subset of  $\mathbb{E}^m$ . It follows by Theorem 2.5 that  $\Delta_n$  is closed and bounded and by Theorem 2.4,  $\alpha_n$ ,  $\beta_n$ , and  $\gamma_n$  have accumulation point in  $\Delta_n$ . Consequently, there exists  $(n_k) \subset \mathbb{N}$  such that  $(A_{n_k})$ ,  $(B_{n_k})$ , and  $(C_{n_k})$  converge to  $A$ ,  $B$ , and  $C$ , respectively. By Lemma 3.1, the points  $A$ ,  $B$ , and  $C$  are all points in  $\Delta_1$ .

Suppose now the contradiction that  $(A_{n_k})$ ,  $(B_{n_k})$ , and  $(C_{n_k})$  do not converge to the same point. Then we consider the following cases.

**Case I.**  $A \neq B \neq C$ . By Corollary 3.2, the vertices of  $\Delta(A, B, C)$  and the innermost triangle  $\Delta'(A, B, C)$  of  $\Delta(A, B, C)$ . Denote by  $d$  the minimum distance between the  $\Delta'$  and each point  $A, B$ , and  $C$ . It follows that  $d/2$ -neighborhoods of  $A$  and  $\Delta'$  are disjoint (likewise for  $B$  and  $C$ ). Since  $(A_{n_k})$ ,  $(B_{n_k})$ , and  $(C_{n_k})$  converge to  $A, B$ , and  $C$ , respectively, there exists  $N \in \mathbb{N}$  such that the  $d/2$ -neighborhood of each edge in the subdivision triangle of  $\Delta$  contains the corresponding edge of  $\Delta(A_{n_N}, B_{n_N}, C_{n_N})$ .

Suppose that  $\Delta_{n_N+1}$  is the inner subdivision of  $\Delta_{n_N}$ . Consequently,  $A$  and  $\Delta_{n_N+1}$  are disjoint. However, it is a contradiction as  $A$  lies in  $\Delta_{n_N+1}$ . If  $\Delta_{n_N+1}$  is not the innermost triangle (but the other triangle  $\Delta(B_{n_N+1}, A_N, C_{n_N+1})$ ), then it is disjoint to either  $B$  or  $C$  which is impossible since  $B$  and  $C$  belong to  $\Delta_{n_N+1}$ .

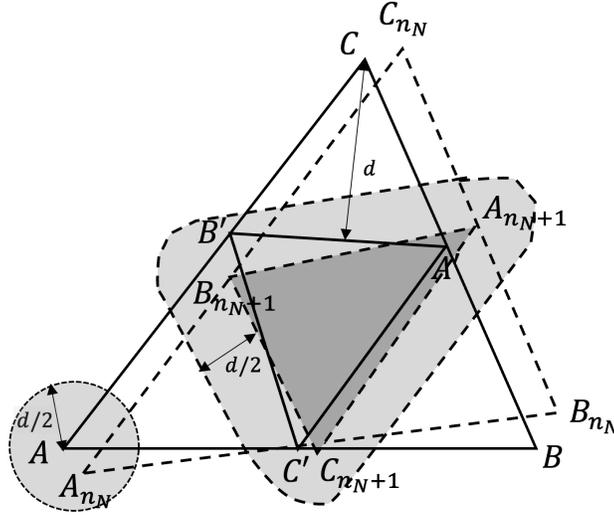


Figure 2. Case 1

**Case 2.**  $A \neq B$  and  $B = C$ . We show this case similar to the first case. Since  $B = C$ , we have a degenerate triangle that the vertices of the innermost triangle  $\Delta(A', B', C')$  of  $\Delta(A, B, C)$  are collinear. In this case  $A'$  coincides with  $B$  and  $C$ . Suppose,  $d = \frac{1}{2} \min\{d(A', B'), d(A', C')\}$ . We then obtain the similar case that  $d/2$ -neighborhood of  $A$  and  $\Delta'$  are disjoint. Following the same as that in case 1, it is impossible. Note that the cases  $A \neq C$  and  $B = C$  and the case  $B \neq C$  and  $A = B$  follow the case  $A \neq B$  and  $B = C$ .

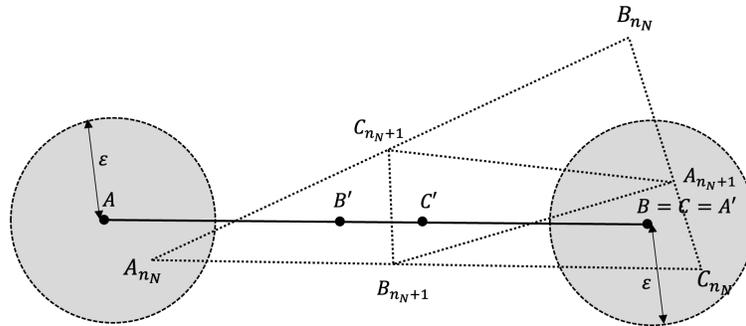


Figure 3. Case 1

From these cases, we find that  $A = B = C$ , or in other words, we say that  $(A_{n_k})$ ,  $(B_{n_k})$ , and  $(C_{n_k})$  converge to the same point. Hence, by the definition of a limit point, the proof is complete.  $\square$

From Theorem 3.3, we have the following lemma.

**Corollary 3.4.** Let  $\Delta_n$  be the iterated Cevian triangle in  $\mathbb{E}^m$ . Then  $\Delta_n$  converges to the  $P = \bigcap_{n \in \mathbb{N}} \text{Int}(\Delta_n)$ .

**Proof.** The existence of  $P$  can be guaranteed as given in the proof of Theorem 3.3. Next, Lemma 3.1 ensures that  $P$  is in the interior of  $\Delta_n$  for all  $n \in \mathbb{N}$ .  $\square$

**Theorem 3.5.** Let  $\Delta$  be a triangle with vertices  $A, B$ , and  $C$  in  $\mathbb{E}^m$  and  $P \in \text{Int} \Delta$ . Then there exists a sequence of iterated Cevian triangles  $\{\Delta_k\}$  with vertices  $A_k, B_k$  and  $C_k$  of  $\Delta$  such that  $\Delta_n$  converges to  $P \in \mathbb{E}^m$ .

**Proof.** Let  $P$  be an arbitrary interior point of  $\Delta$ . Then, draw a line from  $A$  ( $B$  and  $C$ , respectively) passing through  $P$  such that this line cuts  $BC$  ( $AC$  and  $AB$ , respectively) at  $A_1$  ( $B_1$ , and  $C_1$ , respectively). Connecting these points forms the first Cevian triangle  $\Delta_1$  with vertices  $A_1, B_1$ , and  $C_1$  (See Figure 4). Repeating this process, we obtain a sequence of iterated Cevian triangle  $\{\Delta_k\}_{k \in \mathbb{N}}$  with  $P$  is in the interior of  $\Delta_k$  for all  $k \in \mathbb{N}$  by Lemma 3.1. Next, we show that  $\{A_n\}, \{B_n\}$ , and  $\{C_n\}$  are convergent to  $P$ . From Theorem 3.3, we have that  $\{A_n\}, \{B_n\}$ , and  $\{C_n\}$  converge to a unique point in  $\text{Int}(\Delta_n)$ . So, it suffices to show that  $\{A_n\}$  converges to  $P$ .

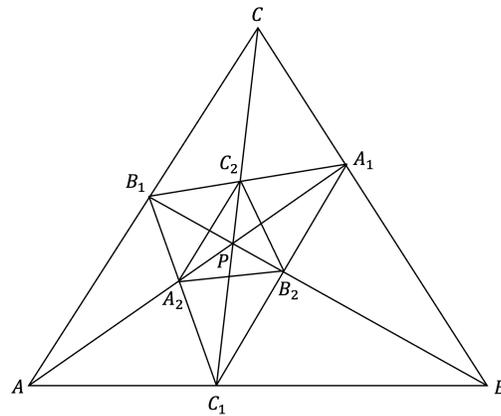
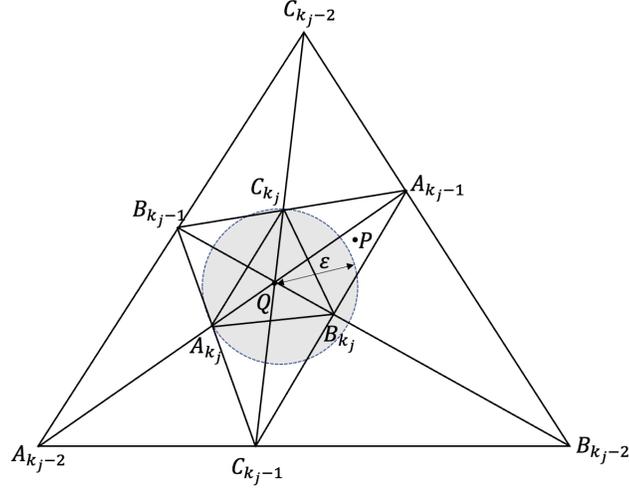


Figure 4. Cevian triangle from point  $P$

Suppose that  $\{A_k\}_{k \in \mathbb{N}}$  converges to  $Q$  with  $Q \neq P$ . Suppose that  $\varepsilon \in \mathbb{R}$  such that  $0 < \varepsilon < d(P, Q)$ . Since  $\{A_k\}_{k \in \mathbb{N}}, \{B_k\}_{k \in \mathbb{N}}$ , and  $\{C_k\}_{k \in \mathbb{N}}$  converge to  $Q$ , there exist  $\{k_i\} \subset \mathbb{N}$  such that maximum distances between the vertices of  $\Delta_{k_j} < \varepsilon$ . It implies that  $P \notin \text{Int}(\Delta_{k_j})$ . It is a contradiction, since  $\Delta_{j_k}$  is formed by vertices from the intersection of the lines passing through the vertices of  $\Delta_{j_k-1}$  and point  $P$ . So  $P = Q$  (see Figure 5).

$\square$



**Figure 5.** Cevian triangle converges to  $Q \neq P$

**Theorem 3.6.** Let  $\{\Delta_k\}$  and  $P$  as illustrated in the Theorem 3.5. Then it satisfies that

$$\lim_{k \rightarrow \infty} \prod_{i=1}^k \left( \frac{PA_{i+1}}{A_i A_{i+1}} + \frac{PB_{i+1}}{B_i B_{i+1}} + \frac{PC_{i+1}}{C_i C_{i+1}} \right) = 1.$$

**Proof.** Consider the triangle  $\Delta_1$  with vertices  $A_1, B_1$ , and  $C_1$  and  $\Delta_2$  with vertices  $A_2, B_2$ , and  $C_2$ , as given in Figure 4. First, we have

$$\begin{aligned} \frac{L\Delta(A_1, B_1, P)}{L\Delta(A_1, B_1, C_1)} &= \frac{L\Delta(A_1, C_2, C_1)}{L\Delta(A_1, B_1, C_1)} \cdot \frac{L\Delta(A_1, C_2, P)}{L\Delta(A_1, C_2, C_1)} \cdot \frac{L\Delta(A_1, C_2, C_1)}{L\Delta(A_1, B_1, C_1)} \\ &= \frac{A_1 C_2}{A_1 B_1} \cdot \frac{P C_2}{C_1 C_2} \cdot \frac{A_1 B_1}{A_1 C_2} \\ &= \frac{P C_2}{C_1 C_2}. \end{aligned}$$

By the same way, we have

$$\frac{L\Delta(B_1, C_1, P)}{L\Delta(A_1, B_1, C_1)} = \frac{P A_2}{A_1 A_2}$$

and

$$\frac{L\Delta(A_1, C_1, P)}{L\Delta(A_1, B_1, C_1)} = \frac{B_2 P}{B_1 B_2}.$$

Combining these results gives us

$$\begin{aligned} \frac{P A_2}{A_1 A_2} + \frac{P B_2}{B_1 B_2} + \frac{P C_2}{C_1 C_2} &= \frac{L\Delta(A_1, C_1, P)}{L\Delta(A_1, B_1, C_1)} + \frac{L\Delta(B_1, C_1, P)}{L\Delta(A_1, B_1, C_1)} + \frac{L\Delta(A_1, B_1, P)}{L\Delta(A_1, B_1, C_1)} \\ &= \frac{L\Delta(A_1, B_1, C_1)}{L\Delta(A_1, B_1, C_1)} \\ &= 1. \end{aligned}$$

Repeating the same process for  $\Delta_i$  and  $\Delta_{i+1}$  for all  $k \in \mathbb{N}$  and multiplying the results yields

$$\prod_{i=1}^k \left( \frac{PA_{i+1}}{A_k A_{i+1}} + \frac{PB_{i+1}}{B_i B_{i+1}} + \frac{PC_{i+1}}{C_i C_{i+1}} \right) = 1.$$

Taking the limit for  $k \rightarrow \infty$  results the desired equation. □

## 4. Conclusions

This study establishes a complete and rigorous framework for the behavior of iterated Cevian triangles in finite Euclidean spaces. We have proven that the iterative process is not only convergent but also surjective: Corollary 3.4 and Corollary 3.2 demonstrate that the sequence of triangles  $\Delta_n$  shrinks uniformly, guaranteeing convergence to a unique interior point  $P$ , while Theorem 3.5 establishes that every point within the original triangle is attainable as such a limit. Finally, Theorem 3.6 provides a profound and distinctive characterization of this convergence by revealing a precise multiplicative invariant—a product of ratios that converges to 1—that is inherently linked to the recursive geometric structure, thereby uncovering a fundamental property unique to the infinite iteration in this discrete setting.

## Acknowledgement

The author would like to express sincere gratitude to the anonymous reviewers for their insightful comments and suggestions, which greatly improved the quality of this manuscript. Helpful discussions with colleagues are also gratefully acknowledged.

## REFERENCES

- [1] K. Seebach, “Ceva-dreiecke,” *Elem. Math*, vol. 42, pp. 132—139, 1987.
- [2] H. Mowaffaq, “The arbitrariness of cevian triangle,” *The American mathematical monthly*, vol. 113, no. 5, pp. 443–447, 2006. <https://doi.org/10.2307/27641951>.
- [3] B. Hvala, “Cevian cousins of a triangle centroid,” *Journal for Geometry and Graphics*, vol. 19, no. 2, pp. 211–218, 2015. <https://www.heldermann-verlag.de/jgg/jgg19/j19h2hval.pdf>.
- [4] Y. N. Aliyev, “Inequalities about the area bounded by three cevian lines of a triangle,” *Elem. Math.*, vol. 80, no. 4, pp. 137—148, 2025. <https://doi.org/10.4171/EM/540>.
- [5] E. Carroll, A. P. Ghosh, X. H. Nguyen, and A. Roitershtein, “Iterated routh’s triangles,” *Journal for Geometry and Graphics*, vol. 21, no. 2, pp. 153–168, 2017. <https://www.heldermann-verlag.de/jgg/jgg21/j21h2carr.pdf>.
- [6] F. Brunck, “Iterated medial triangle subdivision in surfaces of constant curvature,” *Discrete Comput Geom*, vol. 70, pp. 1059–1089, 2023. <https://doi.org/10.1007/s00454-023-00500-5>.
- [7] R. T. Rockafellar, *Convex Analysis*. Princeton University Press, 1970. <https://doi.org/10.1017/S0013091500010142>.
- [8] I. E. Leonard and J. E. Lewis, *Geometry of convex sets*. Wiley, 2016.