Volume 9 No 1 June (2026)

e-ISSN: 2615-3270
p-ISSN: 2615-3599

Eigen Mathematics Journal

homepage: https://eigen.unram.ac.id/index.php/eigen

# Axiomatization of a Fundamental System for Generalized Ambiguous Four–Degree Membership Function Set Theory

**Ali Aji**[1*], **Bello Inuwa Mshelia**[1], **Yusuf Haruna**[1]

[1] *Department of Mathematical Science, Abubakar Tafawa Balewa University, Nigeria*
[*] *Corresponding author:* ali.aji32@gmail.com

## ABSTRACT

This paper presents the construction of an axiomatic system of Generalized Ambiguous Set Theory (AGAST)founded upon a four-valued logic, where the membership relation accepts four discrete degrees: True ($\alpha_A(x)$), Partially True ($\beta_A(x)$), Partially False ($\gamma_A(x)$), and Falsity ($\eta_A(x)$) membership degree functions, The contribution of this work is the rigorous adaptation of foundational Zermelo-Frankel set theory (ZFC) and Axiomatic Fuzzy Set Theory principles, including the Axiom of Extensionality and the Axiom Schema of Ambiguous Separation, to cohere within the four-degree semantics. Furthermore, the theory introduces the Anti-Classicality Axiom, which postulates the existence of sets exhibiting non-classical membership degrees.

**Keywords:** Axiom, Ambiguous, four-degree, set, theory

Received : 13-01-2026;
Revised : 27-02-2026;
Accepted : 11-03-2026;
Published : 23-04-2026;

DOI: https://doi.org/10.29303/emj.v9i1.356

## 1. Introduction

In this paper, we introduce the Axiomatic Theory of Generalized Ambiguous Set (AGAST) as an extension of Axiomatic Fuzzy Set Theory (AFST), intended to serve as a foundational framework for modelling uncertainty in modern mathematics. Classical Zermelo–Fraenkel Set Theory (ZFC), inherently grounded in Boolean logic, restricts propositions to strict bivalence (i.e., True or False). This classical certainty renders ZFC unsuitable for directly modelling phenomena characterized by genuine ambiguity (paraconsistency) and incompleteness (paracompleteness), which are common challenges in advanced computation, artificial intelligence, and control systems. The resulting AGAST framework provides a coherent foundational structure for ambiguous reasoning. Its robustness is demonstrated by its capacity to absorb Russell's paradox through controlled inconsistency rather than exclusion. Structurally, AGAST preserves similarity with ZFC, ensuring that the system constitutes a philosophically sound extension of the Neumann–Bernays–Vaught (NBV) axioms of classical set theory.

The axiom of set theory by [1], after Russel revealed the paradoxes in set theory were coming from lack of proper axiom, Zermelo started working on the axiomatization of the set theory responded to his detractors by publishing a proof of the Well-Ordering Theorem in 1908, he also published the first complete axiomatization of set theory in 1908, recognizing the importance of axiomatization as a universal methodological role in set theory. Neumann–Bernays–Gödel set theory [2] similar to Cantor's work, formed a new mathematical set theory environment rather than a pointless exercise in structural construction. In this instance, the goal was to make the proof more straightforward rather than to formulate and solve a problem. The axiomatic uncertainty, unlike the classical set theory [3]. The [4] axiomatization of fuzzy set theory that depends on its membership degree functions, the standard unit interval [0, 1], and the algebraic operations of sets. With the membership function and the method of constructing membership functions [5]. Research in [6] discussed the new approach of fuzzy axiomatics by establishing a system of axioms of fuzzy sets based on the Zermelo-Fraenkel axioms of set theory, embedding the fuzzy membership degree relation and the set. The study in [7] analyzed the properties and gradual descriptions of logical formulas $(\Gamma)$ that serve as $\tau$-conclusions. It also outlined the step-by-step method (algorithm) for determining the degree of membership for any formula to be a $\tau$-conclusion, based on constructing the theory root within the specified logic systems. A more recent [8] direct use of ZFC as Belnap-Zemermelo (BZFC) axiomatic set theory, constructed entirely within a four-valued paraconsistent and paracomplete logic. BZFC prioritizes a clear ontology of non-classical sets and is derived from a careful translation of the ZFC axioms into the four-valued setting. A key metatheoretical observation regarding systems like BZFC is that they are naturally. The Intuitionistic Fuzzy Set [9] examines the characteristics of these operations using modal and generalized operators. Presenting two tables that show and detail the operational changes in fuzzy sets. The research examined in [10] define various forms of Neutrosophic sets, standard and non-standard Neutrosophic logic, which are analyzed using the three-degree membership. Used in many forms of decision-making and artificial intelligence. Paper on [11] also formulated some crucial definitions, operation, and properties of Generalized neutrosophic set $A$ for all $x \in X$ we have $A = \{x : \mu_A(x), \omega_A(x), \vartheta_A(x),\ x \in X\}$ where the function $\mu, \omega, \vartheta$ define by truth membership $(T)$, indeterminacy $(I)$ and falsity $(F)$ respectively $x \in X$ defined on the interval $]0^-, 1^+[$ and including some important theorems of Generalized neutrosophic set.

The study in [12] formed a constraint that combined the truth and falsity membership to less than or equal to one and created new membership as a contradiction with indeterminacy spanning four degrees of membership, operators, and properties. The research investigated in [13] focuses on four membership degrees as true membership degree $\Pi_t(g)$, false membership degree $\Pi_f(g)$, partially true ambiguous membership degree $\Pi_{ft}(g)$ and partially false ambiguous membership degree $\Pi_{fa}(g)$ are represented for uncertain events in the Ambiguous set theory. In the axiomatization of generalized ambiguous theory, we grade the membership element degree structure and form a four-degree classical set membership. Paper in [14] discussed the generalized ambiguous set theory with four degree membership function as true $\alpha_A(x)$, partially true $\beta_A(x)$, partially false $\gamma_A(x)$, and falsity $\eta_A(x)$ membership degree function, and [15] formalized both fuzzy logic and neutrosophic logic, which incorporates Belnap's four-value logic to form a generalized ambiguous logic.

## 2. Preliminaries

**Definition 2.1.** [14] Generalized Ambiguous Set : Let $A$ be a set defined within a universe $X$. Each element $x$ in $X$ is characterized by four distinct membership functions that map to the interval $[0, 1]$, with truth membership degree function, partially true membership degree function, partially false membership degree function and falsity of membership degree function denoted by $\alpha_A(x)$, $\beta_A(x)$, $\gamma_A(x)$, $\eta_A(x)$ respectively where $0 \leq \langle \alpha_{A_1}(x), \beta_{A_1}(x), \gamma_{A_1}(x), \eta_{A_1}(x) : x \in X \rangle \leq 1^+$.

**Definition 2.2.** [15] Ambiguous Four-Degree Logic is a four-degree membership function which has a true degree membership $\alpha_A$, a partially true degree membership function $\beta_A$, a partially false membership degree function $\gamma_A$, and a falsity degree membership function $\eta_A$, with standard unit interval $[0, 1]$, where $\alpha, \beta, \gamma, \eta \in [0, 1]$ and $0 \leq \inf(\alpha_A + \beta_A + \gamma_A + \eta_A) \leq \sup(\alpha_A + \beta_A + \gamma_A + \eta_A) \leq 1^+$.

**Definition 2.3.** [16] Ambiguous conorms (A-conorms) is an operation $p : (X\bigstar)^2$ on the interval $[0, 1]$, a function $A : ([0, 1] \times [0, 1] \times [0, 1], [0, 1))^2 \rightarrow [0, 1] \times [0, 1] \times [0, 1] \times [0, 1]$ if $A_1 = \langle \alpha_{A1}, \beta_{A1}, \gamma_{A1}, \eta_{A1} \rangle$, $A_2 = \langle \alpha_{A2}, \beta_{A2}, \gamma_{A2}, \eta_{A2} \rangle$, $A_3 = \langle \alpha_{A3}, \beta_{A3}, \gamma_{A3}, \eta_{A3} \rangle$ such that $A(A_1, A_2) = \langle \min(\alpha_1, \alpha_2), \min(\beta_1, \beta_2), \max(\gamma_1, \gamma_2), \max(\eta_1, \eta_2) \rangle$ for $\forall A_1, A_2, A_3 \in A$ A-Conorms satisfied the following conditions:

i. Commutativity: $S(A_1, A_2) = A(A_2, A_1)$
1. Associativity: $S(A_1, A(A_2, A_3)) = A(A_2, A(A_1, A_3))$
ii. Monotonicity: $A(A_1, A_2) \leq A(A_1, A_3)$ if and only if $A_2 \leq A_3$
iii. Boundary Condition: $A(A_1, X^{\bigstar}_{\min}) = A_1$

**Definition 2.4.** [16] Ambiguous-norms (A-norms) is an operation $A : (X^{\bigstar})^2 \rightarrow [0, 1]$ on the interval $[0, 1]$, a function $A : ([0, 1] \times [0, 1] \times [0, 1], [0, 1))^2 \rightarrow [0, 1] \times [0, 1] \times [0, 1] \times [0, 1]$ if $A_1 = \langle \alpha_{A1}, \beta_{A1}, \gamma_{A1}, \eta_{A1} \rangle$, $A_2 = \langle \alpha_{A2}, \beta_{A2}, \gamma_{A2}, \eta_{A2} \rangle$, $A_3 = \langle \alpha_{A3}, \beta_{A3}, \gamma_{A3}, \eta_{A3} \rangle$ such that $A(A_1, A_2) = \langle \max(\alpha_1, \alpha_2), \max(\beta_1, \beta_2), \min(\gamma_1, \gamma_2), \min(\eta_1, \eta_2) \rangle$ for $\forall A_1, A_2, A_3 \in A$ A- norms satisfied the following conditions:

i. Commutativity : $A(A_1, A_2) = A(A_2, A_1)$
ii. Associativity: $A(A_1, A(A_2, A_3)) = A(A_2, A(A_1, A_3))$
iii. Monotonicity : $A(A_1, A_2) \leq A(A_1, A_3)$ if and only if $A_1 \leq A_3$
iv. Boundary Condition: $A(A_1, X^{\bigstar}_{\max}) = A_1$

## 3. Research Methods

To form AGAST, we consider the membership degree functions to create a standard universal model by axioms and the generalized ambiguous logic [15] conventional symbols $(\cup, \cap, \neg, \wedge, \vee, \rightarrow$ and $\leftrightarrow)$ and GAST membership functions: $A = \langle x : \alpha_{A1}(x), \beta_{A1}(x), \gamma_{A1}(x)\eta_{A1}(x) : x \in X \rangle$ where $A(x) : X \longrightarrow x \in [0, 1]$ and $\alpha_A(x)$ (denoted by true membership degree), $\beta_A(x)$ (denoted by partially true membership degree), $\gamma_A(x)$ (denoted by partially false membership degree), and $\eta_A(x)$ (denoted by falsity membership degree), If we assume that $E_\tau(x, y, z, p)$ means $\epsilon_\tau$ as a membership, $x$ as an element of $X$, $y$ as $A$, $z$ as $A(x)$, $\tau$ represents $[0, 1]$, and $\zeta$ as class since $\alpha_{A1}(x), \beta_{A1}(x), \gamma_{A1}(x), \eta_{A1}(x)$ are elements of $A$ (membership degree). Using this formal notation $\forall$ (for all), $\exists$ (there exists), $\varnothing$ (empty set), $\subseteq_4$ (subset), $\notin$ (not belong to), $\in$ (belong to), $\leq_4$ (less than or equal to), $\geq_4$ (greater than or equal to), then we have the following:

**Definition 3.1.** Generalized Ambiguous Set $A$: if the element of the universe $X$ is defined by the four-degree membership as

$$\mu_A = \langle x : \alpha_A(x), \beta_A(x), \gamma_A(x), \eta_A(x) \rangle.$$

**Definition 3.2. the generalized ambiguous lattice $\tilde{L}_G$.** let $\tilde{L}_G$ be the set of quadruplets defined by $\tilde{L}_G = \{\langle \alpha, \beta, \gamma, \eta \rangle \in [0, 1]^4 \mid (\alpha + \beta + \gamma + \eta \leq_4 k)\}$ where bounded constant $(k \leq_4 1)$.

**Definition 3.3.** Generalized Ambiguous Gödel Implication $I_G$: We define the implication as: $I_G : \tilde{L}_G \times \tilde{L}_G \rightarrow \tilde{L}_G$ Component-wise using the Gödel implication $I_G(a, b) = 1$ if $a \leq b$

**Definition 3.4.** the Truth Value: We define a set $U$ and $V$ of four distinct values representing the degrees of Membership using if $A = \langle x : \alpha_{A1}(x), \beta_{A1}(x), \gamma_{A1}(x)\eta_{A1}(x) : x \in X\rangle$ then we define based on the Empty set Let $u_A = \{\varphi, \{\varphi\}, \{\{\varphi\}\}, \{\{\{\varphi\}\}\}\}$ where $\varphi$ for (False $\eta_A(x)$), $\{\varphi\}$ for (Partially False $\gamma_A(x)$), $\{\{\varphi\}\}$ for (Partially True $\beta_A(x)$) and $\{\{\{\varphi\}\}\}$ for (true $\alpha_A(x)$).

To form an axiomatic, ambiguous four-degree membership function set theory (which incorporates truth values of true, partially true, partially false, and false) within the framework of classical set theory, we leverage the axioms of classical set theory, such as Zermelo-Frankel (ZF) set theory. This approach involves representing the membership degrees as a set and then defining the ambiguous sets as functions from a universe to these truth values. Below, I outline the step-by-step construction using classical set theory axioms:

## 4. Result and Discussion

### 4.1. Universe and The Membership Function

Let $U$ be the Universe of discourse, which is a set in classical set theory. An Ambiguous Set $A$ is characterized by a membership Function $\mu_A : U \to V$, which assigns each element $x \in U$ one of the true values. In classical set theory, a function is represented as a set of ordered pairs satisfying the function property (i.e., $x \in U$ there is exactly one $v \in V$ such that $(x, v) \in \mu_A$). The set of all such functions from $U$ to $V$ is denoted by $U^V$, and we identify the Ambiguous Set $A$ with $\mu_A$.

### 4.2. Representing Ambiguous Sets in Classical Set Theory

For each Ambiguous Set $A$;

$$\mu_A = \{(x, y) \mid x \in U, y \in V \text{ and } V = \mu_A(x)\}. \tag{1}$$

This set of ordered pairs is a subset of $U \times V$, where $U \times V$ is the Cartesian product. The function properly ensured that $\mu_A$ is welldefined.

### 4.3. The Core Axioms

The Axiomatization theory of Generalized Ambiguous Set (AGAST) relies primarily on the core axiom, which distinguishes between classical set theory axioms and ensures a meaningful and consistent interpretation. The following axioms are postulated.

**Axiom 4.1.** *(Non-Negativity & Boundedness): The membership degrees are real numbers in the closed unit interval:* $A = \langle x, \alpha_{A1}(x), \beta_{A1}(x), \gamma_{A1}(x), \eta_{A1}(x)\rangle, \quad \forall x \in X,\ A(x) \to x \in [0,1]$ *then* $U \times V \to A(x) \in [0,1]$

**Axiom 4.2.** *(Exhaustive Membership): The sum of all four membership degrees for any element must be 1. This ensures the description of membership is complete:* $\forall x \in X : \alpha_{A1}(x), \beta_A(x), \gamma_A(x), \eta_A(x) = 1$. *We write it in the form of an axiom as* $A(x) = 1$.

**Axiom 4.3.** *(Ordering of Ambiguity): Partial Truth is conceptually "closer" to Truth than to Falsehood, and conversely, Partial Falsehood is "closer" to Falsehood than to Truth. The following constraints impose this:* $\alpha_A(x) \geq_4 \beta_{A1}(x)$ *and* $\gamma_{A1}(x) \leq_4 \eta_{A1}(x)$.

**Axiom 4.4.** *(Independence of Partial States): There is no direct functional dependency between* $\beta_A(x)$ *and* $\gamma_A(x)$. *They are independent, subject only to the constraints of Axioms 4.2 and 4.3. This is the key feature that captures actual ambiguity.*

**Axiom 4.5.** *(Degree): For every element* $x$ *and* $y$*, there exists a specific degree in* $y$ *such that* $\forall x, y \in E_\tau$ *where* $\mu_A(x) \to x \in [0,1]^4$ *which satisfies the conditions:* $\alpha, \beta, \gamma, \eta \in [0,1]$ *and* $\alpha + \beta + \gamma + \eta = 1$.

**Axiom 4.6.** *(Extensionality):*

$$\forall A_1, A_2\ (A_1 = A_2 \iff \forall x\,(A_1(x) = A_2(x))). \tag{2}$$

This axiom ensures the set theory remains extensional, meaning sets are fully determined by their internal membership structure, including their inherited ambiguity and incompleteness. If two sets were permitted to differ only in the ambiguity degree of one element, they would necessarily be distinct under AGAST. This strict requirement prevents the weakening of the system's foundational reach, a concern that arises when extensionality is merely relaxed rather than adapted to the graded context.

**Axiom 4.7.** *(Equality): Let $A_1$ and $A_2$ on the universe $U$ be the two sets in a generalized ambiguous set. They are said to be equal if and only if they have precisely the same four-valued membership extensions of the Generalized Ambiguous lattice given by $\tilde{L}_G = \{\langle \alpha, \beta, \gamma, \eta \rangle \in [0,1]^4 \mid (\alpha + \beta + \gamma + \eta \leq k)\}$ where bounded constant $(k \leq 1)$ and $E(A_1, A_2) \in \tilde{L}_G$. The equality predicate $(=)$ remains crisp (Boolean) on the sets themselves, ensuring that the identity of set objects is well-defined as*

$$(\forall \tilde{L}_G)\ [(x = y) \wedge (x \in \tilde{L}_G) \iff (y \in \tilde{L}_G)]. \tag{3}$$

**Theorem 4.8.** ***(Transitivity of generalized Ambiguous Equality)****: For any generalized ambiguous sets $A_1, A_2, A_3$:*

$$E(A_1, A_2) \wedge E(A_1, A_3) \leq_4 E(A_1, A_3).$$

***Proof*****.** Since the lattice operations are component-wise, it suffices to prove for a single component of $\mu_A$ using the Gödel Implication. We recall the chain rule of Implication: $(a \to b) \wedge (b \to c) \leq (a \to c)$. If $a \in A_1, b \in A_2, c \in A_3$, we check the minimum value of the truth (i.e., that is at least true): $\forall a, b, c \in [0,1] \implies \tilde{L}_G \in [0,1]$, we have to prove by considering the relationship between $a$, $b$, and $c$, and $\min(a \to b, b \to c) \leq (a \to c)$, then the following cases hold:

**Case I:** $a \leq b$ and $b \leq c$, then $a \leq c$
$(a \to b) = 1, \quad (b \to c) = 1 \quad$ then $\quad (a \to c) = 1$ since $\min(1,1) = 1 \leq 1$.

**Case II:** $a \leq b$ and $b > c$, then $a \leq c$ or $a > c$
$(a \to b) = 1, \quad (b \to c) = c \quad$ then $\quad (a \to c) = 1$ since $b > c$. Therefore $\min(1, c) = c$. If $a \leq c$, we have $c \leq 1$. If $a > c$, we have $c \leq c$.

**Case III:** $a > b$ and $b \leq c$.
In this case, both $p$ and $q$ may have any relation $(a \to b) = b$ and $(b \to c) = 1$. If $a \leq c$, we have $b \leq 1$ and If $a > c$, then it is equal to $c$. We must check if $b \leq c$. This is true because our case assumption is $b \leq c$.

**Case IV:** $a > b$ and $b > c$, this implies $a > c$

$$(a \to b) = b, \quad (a \to c) = c$$

$$\min(b, c) = c \quad \text{since } b > c.$$

We conclude that $a > c, (a \to c) = c$. So, $c \leq c$. □

**Axiom 4.9.** *(Null Set): There exists a set, denoted by $\varnothing$, such that no element has a designated membership degree. Formally, let $\mu_A = \{\varphi, \{\varphi\}, \{\{\varphi\}\}, \{\{\{\varphi\}\}\}\}$ where $\varphi$ for (False $\eta_A(x)$), $\{\varphi\}$ for (Partially False $\gamma_A(x)$), $\{\{\varphi\}\}$ for (Partially True $\beta_A(x)$), and $\{\{\{\varphi\}\}\}$ for (true $\alpha_A(x)$). Since there is existence of empty set, the ambiguous four-degree set is now characterized by all possible elements.*

**Axiom 4.10.** *(Pairing): Given any two sets* $A_1$ *and* $A_2$*, there exists a set* $\{A_1, A_2\}$ *whose only elements are those of* $A_1$ *and* $A_2$*. This axiom is structurally similar to its ZFC counterpart, focusing on the existence of the container set defined over the four-valued domain. That is* $\forall x[x \in A_1$ *and* $x \in A_2]$*. If* $\mu_{A1} = \{(x, v) \mid x \in V, y \in U\}$ *and* $\mu_{A2} = \{(x, v) \mid x \in V, y \in U\}$ *then* $U \times V$ *is a pairing which forms the equation* $\mu_A = \{(\mu_{A1}, \mu_{A2}) \mid x \in V, y \in U\}$ *defined as*

$$\forall x \forall y \exists p[(p \in \mu_A) \iff ((p = x) \vee (p = y))]. \tag{4}$$

**Axiom 4.11.** *(Generalized Ambiguous Separation): For any ambiguous set* $A$ *and any formula* $\phi(x)$*, there exists a set* $V$ *such that for every element* $x$*, the membership degree of* $x$ *in* $V$ *is defined by the four-valued conjunction of* $x$*'s original membership in* $A$ *and the truth degree of the defining predicate* $\phi(x)$*.*

$$\forall A \forall V \forall x[V(x) = A(x) \wedge (\phi(x))]. \tag{5}$$

This approach ensures that set formation is intrinsically graded. If $x$ is classically true in $A$ ($A(x) = \alpha(x)$), but the predicate $\phi(x)$ is ambiguous (since $\mu_A = \{\varphi, \{\varphi\}, \{\{\varphi\}\}, \{\{\{\varphi\}\}\}\}$), then $x$'s membership in the resulting subset $V$ becomes ambiguous. This strategy of restriction through graded conjunction controls the power of comprehension, aligning with the necessity to restrict the definitional formula $\phi$ in non-classical set theories to avoid immediate inconsistency.

**Axiom 4.12.** *(Union): For any set* $A$*, there exist sets* $A_1$ *and* $A_2$ *(the union of the elements of* $A_1$ *and* $A_2$*) such that the membership degree of any element* $x$ *in* $A$ *is the disjunction (A-norm) of* $x$*'s membership degrees across all sets contained in* $A$*.*

**Definition 4.1.** Generalized ambiguous proper subset ($\subseteq_4$): If $A_1$ and $A_2$ are two generalized ambiguous sets, then $A_1 \subseteq_4 A_2$ if and only if $A_1 = A_2 \iff [A_1 \subseteq_4 A_2] \wedge [A_2 \subseteq_4 A_2]$.

**Axiom 4.13.** *(Power Set): For every set* $A$*, there exists a set* $P(A)$ *as the power of the generalized ambiguous set* $A_1$*, whose elements are all the proper subsets of* $A_1$*. If there exists* $P(A) = [x : A_1 \mid A_1 \subseteq_4 A]$*, where* $A_1 \subseteq_4 A$ *with the inclusion conditions* $\alpha_{A2} \leq \alpha_{A1}$*,* $\beta_{A2} \leq \beta_{A1}$*,* $\gamma_{A2} \geq \gamma_{A1}$*,* $\eta_{A2} \geq \eta_{A1}$*, and* $(A\neg\neg_1) \in \tilde{L}_G$*, we define the ambiguous power set* $P(A)$ *as:*

$$\forall x[x \in \tilde{L}_G \iff P(A)]. \tag{6}$$

**Corollary 4.14.** *If* $A_1 \subseteq_4 A_2$ *then* $P(A_1) \subseteq_4 P(A_2)$

***Proof.*** Let $S \in P(A)$. By Definition 4.1, $S \subseteq_4 A_1$ and we have $A_1 \subseteq_4 A_2$,
($\Rightarrow$) $\alpha_{A2} \leq \alpha_{A1}$, $\beta_{A2} \leq \beta_{A1}$, $\gamma_{A2} \geq \gamma_{A1}$, $\eta_{A2} \geq \eta_{A1}$. Then $S \subseteq_4 A_1$ and $A_1 \subseteq_4 A_2$. Therefore $S = \langle \alpha_A, \beta_A, \gamma_A, \eta_A \rangle \in [0, 1]$, $A_1 \in [0, 1]$.
($\Rightarrow$) $S \subseteq_4 A_2$. Therefore $S \in P(A)$ satisfying the condition. $P(A_1) \subseteq_4 P(A_2)$ □

**Axiom 4.15.** *(Schema of Ambiguous Replacement): The Schema of Ambiguous Replacement asserts that if for all* $\phi(x, y)$*, if* $A_1 = \{\langle x, \delta_A(x) \rangle \mid x \in X\}$*, then we have* $\psi(x, \delta_y)$ *define a:*

$$\forall x \exists \delta_y \, \psi(x, \delta_y) \Rightarrow \forall A_1 \exists A_2 \forall \delta_y \left( \delta_y \in A_2 \Leftrightarrow \exists x A_1 \, \psi(x, \delta_y) \right). \tag{7}$$

**Axiom 4.16.** *(Generalized Ambiguous Regularity): The Axiom of Regularity (Foundation) prevents infinite descending the existence of sets that are members of themselves* ($x \in A$)*. In AGAST, the existence of non-classical sets necessitates a modified four-degree membership set. While classical regularity must hold for classical subsets (those whose memberships are restricted to* $\{$*True and False*$\}$*),* *the non-classical nature of AGAST must permit self-membership to exist with degree partially true or partially false without collapsing the system, thereby tolerating non-well-boundedness in the ambiguous domain, similar to how non-well-founded sets are treated in other extensions.*

$$\forall A\,[(\exists x \mu_A(x) > 0) \to \exists y\,(\mu_A(y) > 0 \wedge \forall z(\mu_A(z) > 0 \to (\mu_A(z) = 0)))] \tag{8}$$

**Corollary 4.17.** *Existence of an Ambiguous set:* $\forall U \exists A(U)$

***Proof.*** By the Definition of the Ambiguous Set, this shows that we have $\exists v$ and $\forall U \exists (U \times V)$ for any empty set $\phi \in A(U)$ such that $\Psi_\phi(x) \quad \forall x \in U$. This indicates the existence of complete membership elements that lie within the interval $[0,1]$. For any elements $x_0 \in U$ and any membership grade $\tau \in A$, $\exists A(x_0, \tau) \in A(U)$ such that

$$\psi_\phi(x_0, \tau) = \begin{cases} 1, & x = x_0 \\ 0, & x \neq x_0 \end{cases}.$$

For each $x \in U$, $\exists \psi_\phi(x_0, \tau)(x) \in A \Rightarrow A(U) \to x \in [0,1]$. Defines the existence membership functions in the universe $U$. □

## 5. Conclusions

The primary metatheoretical concern for any set theory is consistency. Classical set theories must strictly prohibit contradictions because of the principle of explosion. Since AGAST is constructed on the four-valued logic, which is intrinsically paraconsistent, it invalidates the principle of explosion. Local contradictions, such as an element x having membership degree in set A, do not lead to the immediate trivialization of the entire theory. The framework was established by demonstrating its relative consistency with ZFC. If ZFC is consistent, then AGAST is also consistent. This relies on the bi-interpretability where AGASTsets can be modelled within a ZFC universe as ordered pairs of classical sets representing the set's extension and anti-extension. This approach confirms that the philosophical decision to treat ambiguity and incompleteness as fundamental degrees of freedom does not introduce uncontrollable logical instability. The handling of any foundational set theory ZFC avoids this by ensuring the collection defined by the property "not a member of itself" is not a set under the Axiom of Separation. In AGAST, the Axiom of Ambiguous Separation still restricts comprehension, ensuring that the defining property yields a graded result. When the statement is evaluated within a degree membership function, it does not resolve to a classical true or false, but rather to the states of true, partially true, partially false, and false. The set exists, and its self-membership is non-classical, meaning that AGAST is paraconsistent, tolerating the designated values, which preserves non-triviality.

Finally, this work successfully establishes the axiomatic system AGAST for a set theory based on a Four-valued paraconsistent and paracomplete membership degree domain ($A = \langle x : \alpha_{A1}, \beta_{A1}, \gamma_{A1}, \eta_{A1} : x \in X \rangle$). The system relies on the algebraic structure of four-value logic derived from Belnap's logic, which defines graded logical connectives that tolerate contradiction and indeterminacy. Key axiomatic departures from ZFC include the Axiom of Extensionality, which defines equality based on four-valued characteristic functions, and the Axiom of Ambiguous Separation, which utilizes four-valued conjunction to grade the formation of subsets. The ontological significance of AGAST provides a coherent, foundational means of integrating contradictory and incomplete information into the mathematical domain, demonstrating its power by forming a well-defined property that sufficiently satisfies an ambiguous ontological entity.

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