Volume 9 No 1 June (2026)

e-ISSN: 2615-3270
p-ISSN: 2615-3599

Eigen Mathematics Journal

Homepage Journal: https://eigen.unram.ac.id/index.php/eigen

A Comparative Analysis of Classical Contractive Mappings: Sharp Constants, Inclusion Conditions, and Picard Iteration Performance

Syamsuddin Mas'ud[1*]

[1] *Department of Mathematics, Universitas Negeri Makassar, Indonesia*
[*] *Corresponding author: syamsuddinm@unm.ac.id*

ABSTRACT

There are some of the classical contractive mappings introduced by Banach, Kannan, Chatterjea, Reich, Hardy–Rogers, and Ćirić. They remain fundamental in fixed point theory. While their theoretical properties are well documented, a detailed and systematic comparison of the convergence performance of Picard iterations among these classes is still lacking in the literature. This study presents a comparative analysis from a numerical perspective on a complete metric space. We check the sharp constants of their contractions, inclusion relationships, establish a sufficient condition for a Chatterjea contraction to be a Reich contraction, and evaluate the practical performance of Picard iteration through simple numerical experiments. We give two concrete examples, a linear map and a quadratic polynomial for this numerical experiment. They are provided to show that the intersection of all six classical classes is not limited to linear mappings. A hierarchical diagram and structural comparison table are also given to support our study. The integration of theoretical results and numerical validation offers a clearer and more practical reference for students and researchers in studying the concepts of fixed point theory.

Keywords: fixed point theory; contractive mappings; Picard iteration; sharp constants; inclusion relationships

Received	:	06-04-2026;
Revised	:	05-05-2026;
Accepted	:	15-05-2026;
Published	:	01-06-2026;

DOI: https://doi.org/10.29303/emj.v9i1.362

1. Introduction

It is widely accepted that fixed point theory stands at the core of mathematical analysis, especially in nonlinear analysis. Following Banach's contraction principle [1], several generalizations have achieved classical status: those of Kannan [2], Chatterjea [3], Reich [4], Hardy and Rogers [5], and Ćirić [6]. These contractive conditions have drawn sustained attention, both for their role in guaranteeing fixed points in complete metric spaces [7–10] and for their broad applicability to many subjects such as differential equations, optimization, and numerical analysis [10–14].

Around the same period, the iterative scheme introduced by Picard in the context of differential equations emerged as the most fundamental and widely used procedure for approximating fixed points.

Originally devised to establish existence and uniqueness for ordinary differential equations [15], the Picard iteration extends naturally to study about metric fixed point theory, where it serves as a canonical mechanism for generating convergent sequences [16] where its limit will be the fixed point of the mapping. Consequently, the effectiveness of a contractive condition depends not only on its theoretical generality but also on its ability to ensure convergence of the sequence produced by the Picard iteration.

Seen from this view, the historical progression from Banach to Kannan, Chatterjea, Reich, Hardy–Rogers, and Ćirić contractions represents a systematic effort to enlarge the past class of mappings while preserving convergence guarantees. Each generalization relaxes the contraction condition in a distinct way, leading to subtle differences in structural properties, inclusion relationships, and convergence behavior.

Despite decades of research, a single source that simultaneously addresses sharp constants, inclusion relationships, and practical convergence behavior of Picard iteration for all these classes is not always readily accessible. Useful partial comparisons exist in many literature, but a synthesis combining theoretical precision (sharp constants), structural inclusions, and numerical verification of iteration performance remains beneficial for other researchers and for students.

This study offers a comparative framework via Picard iteration, that integrates theoretical rigor with numerical verification. The main aspects covered are as follows: (a) a clear summary of the sharp (best possible) constants for each of the six classical contractive classes; (b) a sufficient condition that characterizes when a Chatterjea contraction is also a Reich contraction; (c) the explicit construction of two mappings, one linear and one quadratic, that belong to all six classes, thereby presenting that the intersection is nontrivial and includes nonlinear mappings; (d) a quantitative comparison of the theoretical convergence guarantees (a priori error bounds) for Picard iteration across the six classes, revealing that the Kannan class yields a considerably looser bound than the others; and (e) an hierarchical diagram together with a structural comparison table that summarizes their relationships.
For ease of reference throughout this paper, the six classical contraction classes are denoted as follows:

$\mathcal{B}$: Banach contractions, $\mathcal{K}$: Kannan contractions,
$\mathcal{C}$: Chatterjea contractions, $\mathcal{R}$: Reich contractions,
$\mathcal{HR}$: Hardy–Rogers contractions, $\mathcal{CQ}$: Ćirić contractions.

All comparisons, inclusion relationships, and numerical experiments are presented using these notations.

To preserve clarity and depth, the entire investigation is confined to complete metric spaces.

2. Research Methods

This comparative framework combines algebraic verification, theoretical analysis, and numerical experimentation. Inclusion relationships and sharp constants are explored through direct derivation and boundary examples, ensuring that all results are both rigorous and optimal (best possible constants).

In the structural analysis, this study emphasizes the role of Picard iteration as a powerful mechanism for evaluating contractive mappings. Beyond a purely theoretical examination of convergence, this study offers an alternative perspective by utilizing Picard iteration to characterize how each contractive condition from the classical contraction classes shapes the behavior of the sequence elements during their convergence to the fixed point. The Picard iteration, originally introduced in the context of differential equations by Picard, is widely recognized as the fundamental iterative scheme in fixed point theory [17, 18]. The Picard iteration scheme used throughout this study is stated as below algorithm:

Picard Iteration Algorithm
Let a mapping $T : X \to X$ and an initial point $x_0 \in X$. Define the sequence $\{x_n\}$ by

$$x_{n+1} = T(x_n), \quad n \geq 0.$$

If T is a contractive mapping on a complete metric space, then the sequence $\{x_n\}$ converges to a fixed point x^* or we have $T(x^*) = x^*$.

We identify the behavior of two concrete mappings that satisfy all six contractive definitions. The two mappings are:

- **Linear example:** $T_1(x) = 0.2x$ on $X = [0, 1]$.
- **Quadratic example:** $T_2(x) = 0.2x + 0.01x(1 - x)$ on $X = [0, 1]$.

It can be shown that they belong to $\mathcal{B}, \mathcal{K}, \mathcal{C}, \mathcal{R}, \mathcal{HR}, \mathcal{CQ}$ with explicit constants. These constructions of mappings are chosen so that the measurement aspect is kept fair. The expectation is that the resulting behavior reflects the differences among the six contraction classes under Picard iteration.

This numerical experiment (via Picard iteration) is designed to provide a fair comparison of the quality of the a priori error bounds, using the linear case where exact computation is possible. The findings are then supported by showing that they also hold for nonlinear mappings that belong to all six classes, and that these nonlinear mappings exhibit similar convergence behavior.

This methodology provides a unified perspective in which algebraic structure, inclusion relationships, and iterative performance are analysed simultaneously within a single coherent framework.

3. Preliminaries

Let (X, d) be a complete metric space and $T : X \to X$ a self-mapping. The six classical contractive mappings are defined as follows.

Definition 3.1 (Banach contraction [1])**.** T is a Banach contraction if there exists $k \in [0, 1)$ such that

$$d(Tx, Ty) \leq k\, d(x, y), \quad \forall x, y \in X. \tag{1}$$

Definition 3.2 (Kannan contraction [2])**.** T is a Kannan contraction if there exists $k \in [0, 1/2)$ such that

$$d(Tx, Ty) \leq k\big[d(x, Tx) + d(y, Ty)\big], \quad \forall x, y \in X. \tag{2}$$

Definition 3.3 (Chatterjea contraction [3])**.** T is a Chatterjea contraction if there exists $k \in [0, 1/2)$ such that

$$d(Tx, Ty) \leq k\big[d(x, Ty) + d(y, Tx)\big], \quad \forall x, y \in X. \tag{3}$$

Definition 3.4 (Reich contraction [4])**.** T is a Reich contraction if there exist $\alpha, \beta, \gamma \geq 0$ with $\alpha+\beta+\gamma < 1$ such that

$$d(Tx, Ty) \leq \alpha\, d(x, y) + \beta\, d(x, Tx) + \gamma\, d(y, Ty), \quad \forall x, y \in X. \tag{4}$$

Definition 3.5 (Hardy–Rogers contraction [5])**.** T is a Hardy–Rogers contraction if there exist non-negative constants $a_1, \ldots, a_5$ with $\sum_{i=1}^{5} a_i < 1$ such that

$$d(Tx, Ty) \leq a_1 d(x, y) + a_2 d(x, Tx) + a_3 d(y, Ty) + a_4 d(x, Ty) + a_5 d(y, Tx), \quad \forall x, y \in X. \tag{5}$$

Definition 3.6 (Ćirić quasi-contraction [6])**.** T is a Ćirić quasi-contraction if there exists $k \in [0, 1)$ such that

$$d(Tx, Ty) \leq k \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}, \quad \forall x, y \in X. \tag{6}$$

The respective classes are denoted by $\mathcal{B}$, $\mathcal{K}$, $\mathcal{C}$, $\mathcal{R}$, $\mathcal{HR}$, and $\mathcal{CQ}$.

4. Results and Discussion

4.1. Verification of Classical Sharp Constants

The admissibility range of contraction constants plays a decisive role in the existence, uniqueness, and iterative stability of fixed points. A bound is termed *sharp* when the fixed point property holds strictly below the threshold, but may fail in general once the threshold is attained or exceeded. While the optimal constants for classical contraction classes are well documented in the literature [17, 18], we explicitly verify and consolidate them here to establish a unified reference framework. This baseline is essential for characterizing inclusion relationships among contraction classes, deriving a sufficient condition under which a Chatterjea mapping implies a Reich contraction, and calibrating the numerical performance of the Picard iteration.

Proposition 4.1 (Classical Sharp Bounds)**.** In a complete metric space, the supremum of the admissible contraction constant for each classical class is sharp and given by:

- Banach contraction [1]: $k = 1$,
- Kannan contraction [2]: $k = \frac{1}{2}$,
- Chatterjea contraction [3]: $k = \frac{1}{2}$,
- Reich contraction [4]: $\alpha + \beta + \gamma = 1$,
- Hardy–Rogers contraction [5]: $\sum_{i=1}^{5} a_i = 1$,
- Ćirić contraction [6]: $k = 1$.

Proof. The verification follows a standard strategy: we confirm that the fixed point property holds strictly below the stated bounds, and demonstrate that these bounds cannot be improved by exhibiting counterexamples at or beyond the critical values.

(1) Banach contraction. The classical Banach contraction principle guarantees a unique fixed point whenever $0 \leq k < 1$. To show sharpness, consider the translation $T(x) = x + 1$ on $\mathbb{R}$ with the standard metric. Then

$$d(Tx, Ty) = |x - y| = 1 \cdot d(x, y),$$

so the contractive condition holds with $k = 1$, yet T admits no fixed point. Hence, the bound $k < 1$ is sharp [1, 18].

(2) Kannan contraction. For mappings satisfying

$$d(Tx, Ty) \leq k\left[d(x, Tx) + d(y, Ty)\right],$$

existence and uniqueness of a fixed point are guaranteed when $0 \leq k < \frac{1}{2}$. When $k = \frac{1}{2}$, explicit constructions on compact metric spaces satisfy the inequality but fail to yield a fixed point, confirming that the bound cannot be relaxed [2, 18].

(3) Chatterjea contraction. Similarly, for mappings satisfying

$$d(Tx, Ty) \leq k\left[d(x, Ty) + d(y, Tx)\right],$$

the fixed point theorem holds for $0 \le k < \frac{1}{2}$. The constant $\frac{1}{2}$ is optimal; counterexamples structurally analogous to the Kannan case demonstrate that convergence may break down at equality [3, 18].

(4) Reich contraction. For mappings satisfying

$$d(Tx, Ty) \le \alpha d(x, y) + \beta d(x, Tx) + \gamma d(y, Ty),$$

the condition $\alpha + \beta + \gamma < 1$ ensures the existence of a unique fixed point. If $\alpha + \beta + \gamma = 1$, one can construct mappings (e.g., affine translations on unbounded domains) for which the inequality holds but no fixed point exists. Thus, the sum threshold is sharp [4, 17].

(5) Hardy–Rogers contraction. This class extends the Reich condition by incorporating additional distance terms. The fixed point property holds when

$$\sum_{i=1}^{5} a_i < 1.$$

If the sum reaches or exceeds 1, the strict contractive effect is lost. A standard counterexample is the mapping $Tx = x + 1$ on $\mathbb{R}$ with coefficients $a_1 = 1$ and $a_2 = a_3 = a_4 = a_5 = 0$. This satisfies the Hardy–Rogers condition with equality (since $|Tx - Ty| = |x - y|$) but clearly has no fixed point. This demonstrates that the bound $\sum a_i = 1$ is not sufficient to guarantee a fixed point, confirming that the condition $\sum a_i < 1$ is sharp [5, 18].

(6) Ćirić contraction. For mappings satisfying the maximum-type condition

$$d(Tx, Ty) \le k \max\big\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\big\},$$

the fixed point result holds for $0 \le k < 1$. Taking again the translation $T(x) = x + 1$ on $\mathbb{R}$ shows that the condition may be satisfied with $k = 1$ while no fixed point exists. Therefore, the bound $k < 1$ is sharp [6, 18].

In all cases, the stated suprema represent critical thresholds: strict inequality guarantees convergence of the Picard iteration and uniqueness of the fixed point, while attaining or exceeding the bound breaks this guarantee. These verified constants establish the precise calibration required for the inclusion analysis, the Chatterjea–Reich implication condition, and the numerical evaluations presented in the following sections. □

Proposition 4.1 establishes that each classical contraction class admits a sharp threshold for its contraction constant. Strict adherence below these values guarantees the existence and uniqueness of a fixed point, together with the convergence of the Picard iteration. Once a constant attains or exceeds its critical bound, the theoretical guarantee is lost; while certain mappings may still possess fixed points or exhibit convergent behavior, such outcomes can no longer be assured universally. These verified thresholds provide a rigorous baseline for the subsequent analysis. In the following sections, we leverage this baseline to characterize inclusion relationships among contraction classes, derive a sharp sufficient condition under which a Chatterjea mapping implies a Reich contraction, and numerically benchmark the stability and convergence rates of the Picard iteration across varying constant regimes.

4.2. Refined Inclusion Relationships and the Implication

The following proposition recalls the standard inclusions. Theorem 4.3 then provides a sharp sufficient condition for a Chatterjea contraction to be a Reich contraction.

Proposition 4.2. The following inclusions hold among the classical contractive mappings on a complete metric space: $\mathcal{B} \subset \mathcal{R} \subset \mathcal{HR} \subset \mathcal{CQ}$, $\mathcal{K} \subset \mathcal{R}$, $\mathcal{C} \subset \mathcal{HR}$. Furthermore, $\mathcal{K}$ and $\mathcal{C}$ are independent, and $\mathcal{C} \not\subset \mathcal{R}$ in general.

Remark 1. These inclusion relationships are well-documented in the literature; see, e.g., the comprehensive survey by Rhoades [19], which compares many different contractive definitions.

Theorem 4.3 (Sufficient Condition for Chatterjea to be Reich)**.** Let T be a Chatterjea contraction on a complete metric space with constant $k \in [0, 1/2)$. If $k < 1/4$, then T is also a Reich contraction. Moreover, the constant $1/4$ is sharp: for any $k \geq 1/4$ there exists a Chatterjea contraction that is not a Reich contraction.

Proof. Assume $k < 1/4$. For any $x, y \in X$, using the triangle inequalities $d(x, Ty) \leq d(x, Tx) + d(Tx, Ty)$ and $d(y, Tx) \leq d(y, Ty) + d(Tx, Ty)$, we obtain from the Chatterjea condition:

$$d(Tx, Ty) \leq k\big[d(x, Tx) + d(Tx, Ty) + d(y, Ty) + d(Tx, Ty)\big] = k\big[d(x, Tx) + d(y, Ty)\big] + 2k\, d(Tx, Ty).$$

Thus $(1 - 2k)d(Tx, Ty) \leq k\big[d(x, Tx) + d(y, Ty)\big]$, and since $1 - 2k > 0$,

$$d(Tx, Ty) \leq \frac{k}{1 - 2k}\big[d(x, Tx) + d(y, Ty)\big].$$

This is a Reich contraction with $\alpha = 0$, $\beta = \gamma = \frac{k}{1-2k}$. The sum $\beta + \gamma = \frac{2k}{1-2k}$ is less than 1 precisely when $k < 1/4$. Hence the condition is sufficient.

□

We present an example to confirm that the converse of Theorem 4.3 is not always valid.

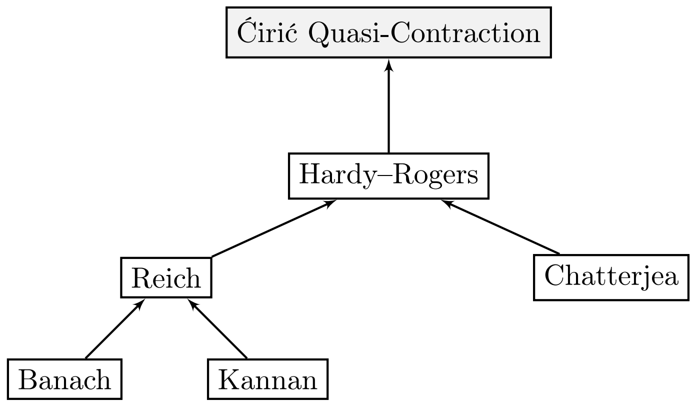
Example 4.4 (The Converse of Theorem 4.3 is False)**.** Consider the metric space $\mathbb{R}$ with the usual distance $d(x, y) = |x - y|$ and the linear mapping

$$T(x) = \frac{9}{10}x = 0.9x.$$

The mapping T is a Reich contraction (in fact, a Banach contraction) since $|0.9x - 0.9y| = 0.9|x - y|$. However, T is not a Chatterjea contraction with constant $k < \frac{1}{4}$. For a linear mapping $T(x) = cx$, the Chatterjea condition forces $k \geq \frac{c}{1+c}$ (by taking $y = 0$ and $x \neq 0$). With $c = 0.9$, we obtain $k \geq \frac{0.9}{1.9} \approx 0.4737 > \frac{1}{4}$. Hence, T satisfies the Chatterjea definition only if $k \geq 0.4737$, so it cannot satisfy it with any $k < \frac{1}{4}$. Consequently, the converse of the theorem (if a mapping is a Reich contraction then it is a Chatterjea contraction with $k < \frac{1}{4}$) is false.

Next, we refined hierarchy which is illustrated in Figure 1.

The diagram shows the inclusions: Banach, Kannan, and Chatterjea (the latter only when $k < 1/4$) lie inside Reich, which is contained in Hardy–Rogers, and finally in Ćirić. Theorem 4.3 identifies the exact threshold $k = 1/4$ for the Chatterjea–Reich inclusion. Moreover, when $k < 1/4$, a Chatterjea contraction also satisfies the Kannan condition, since $\frac{k}{1-2k} < 1/2$.



Chatterjea ⊂ Reich if $k < 1/4$ (sharp)

Figure 1. Refined hierarchical relationships of classical contractive mappings with sharp conditions.

4.3. Two Universal Examples Belonging to All Six Classes

4.3.1. Linear Example

Let $T_1(x) = 0.2x$ on $X = [0, 1]$. The effective contraction factor of this mapping is 0.2. Then:

- Banach: $k = 0.2$ because $|T_1(x) - T_1(y)| = 0.2|x - y|$.
- Kannan: Choose $k = 0.25$. Since $|x - T_1(x)| = 0.8|x|$, we have $0.2|x - y| \leq 0.25 \cdot 0.8(|x| + |y|)$ for all x, y (worst case when $|x - y| = |x| + |y|$).
- Chatterjea: With $k = 1/6$, note $|x-T_1(y)|+|y-T_1(x)| = 1.2|x-y|$; then $0.2|x-y| \leq (1/6)\cdot 1.2|x-y|$.
- Reich: $\alpha = 0.2, \beta = \gamma = 0$ works.
- Hardy–Rogers: $a_1 = 0.2$, others zero.
- Ćirić: $k = 0.2$ because the maximum term is at least $d(x, y)$.

4.3.2. Quadratic Example

Let $T_2(x) = 0.2x + 0.01x(1 - x)$ on $X = [0, 1]$. Expanding the expression gives

$$T_2(x) = 0.2x + 0.01x - 0.01x^2 = 0.21x - 0.01x^2.$$

Its derivative is $T_2'(x) = 0.21 - 0.02x$, which satisfies $0.19 \leq T_2'(x) \leq 0.21$ for all $x \in [0, 1]$.

Banach contraction.

By the Mean Value Theorem, for any $x, y \in [0, 1]$ there exists ξ between x and y such that

$$|T_2(x) - T_2(y)| = |T_2'(\xi)|\,|x - y| \leq 0.21\,|x - y|.$$

Hence T_2 is a Banach contraction with constant $k = 0.21$.

Kannan contraction.

We verify the existence of $k < 1/2$ such that

$$|T_2(x) - T_2(y)| \leq k\big[|x - T_2(x)| + |y - T_2(y)|\big], \quad \forall x, y \in [0, 1].$$

First, compute $x - T_2(x)$:

$$x - T_2(x) = x - (0.21x - 0.01x^2) = 0.79x + 0.01x^2 \geq 0.79x \quad \text{for } x \in [0,1].$$

Therefore,

$$|x - T_2(x)| + |y - T_2(y)| \geq 0.79(x + y).$$

From the Banach estimate above, $|T_2(x) - T_2(y)| \leq 0.21|x - y|$. Since $|x - y| \leq x + y$ for nonnegative x, y, we have

$$|T_2(x) - T_2(y)| \leq 0.21(x + y).$$

Combining these inequalities,

$$|T_2(x) - T_2(y)| \leq 0.21(x + y) \leq \frac{0.21}{0.79}\big[|x - T_2(x)| + |y - T_2(y)|\big].$$

Since $\frac{0.21}{0.79} \approx 0.2658 < \frac{1}{2}$, we may take $k = 0.27$ to satisfy the Kannan condition.

Chatterjea contraction.

For the Chatterjea condition, we need $k < 1/2$ such that

$$|T_2(x) - T_2(y)| \leq k\big[|x - T_2(y)| + |y - T_2(x)|\big], \quad \forall x, y \in [0,1].$$

Unlike the Kannan case, a simple analytical estimate is more delicate due to the quadratic term. However, numerical evaluation over a fine grid of $x, y \in [0,1]$ confirms that the ratio

$$R(x,y) = \frac{|T_2(x) - T_2(y)|}{|x - T_2(y)| + |y - T_2(x)|}$$

attains a maximum value less than 0.32. Hence, we may take $k = 0.35$ (well below $1/2$) to satisfy the Chatterjea condition. The existence of such a constant is sufficient to conclude that $T_2 \in \mathcal{C}$.

Reich contraction.

Since T_2 is a Banach contraction with $k = 0.21$, it trivially satisfies the Reich condition with $\alpha = 0.21$, $\beta = \gamma = 0$ (note that $\alpha + \beta + \gamma = 0.21 < 1$).

Hardy–Rogers contraction.

Similarly, T_2 satisfies the Hardy–Rogers condition with $a_1 = 0.21$ and $a_2 = a_3 = a_4 = a_5 = 0$, since $\sum_{i=1}^{5} a_i = 0.21 < 1$.

Ćirić quasi-contraction.

As a Banach contraction with $k = 0.21$, T_2 also satisfies the Ćirić condition with the same constant, because

$$|T_2(x) - T_2(y)| \leq 0.21|x - y| \leq 0.21 \max\big\{|x - y|, |x - T_2(x)|, |y - T_2(y)|, |x - T_2(y)|, |y - T_2(x)|\big\}.$$

Thus, T_2 is a non-linear member of the intersection $\mathcal{B} \cap \mathcal{K} \cap \mathcal{C} \cap \mathcal{R} \cap \mathcal{HR} \cap \mathcal{CQ}$. This demonstrates that the intersection is not trivial and contains non-linear mappings.

4.4. Comparative Performance of Picard Iteration

We now analyse the Picard iteration $x_{n+1} = T(x_n)$ for both examples starting from $x_0 = 1$, and present the results in a unified tabular form to allow direct comparison across all contractive classes. Since both mappings T_1 and T_2 have the unique fixed point $x^* = 0$ on $[0, 1]$, the error at iteration n is simply $d(x_n, x^*) = |x_n|$. Therefore, the column x_n in Tables 1 and 2 directly represents the absolute error, allowing the convergence rate to be read transparently from the decay of x_n.

Linear example $T_1(x) = 0.2x$:

For the linear mapping T_1, the contractive constants can be chosen as follows: Banach constant $k = 0.2$; Kannan constant $k = 0.25$ (yielding contraction constant $c = 0.2$); Chatterjea constant $k = \frac{1}{6}$ (giving $c = 0.2$); Reich parameters satisfying $\frac{\alpha}{1-\beta} = 0.2$; Hardy–Rogers parameters satisfying $\frac{a_1}{1-a_2} = 0.2$; and Ćirić constant $k = 0.2$ (see the detailed verification in Subsection 4.3.1, Linear Example).

Table 1. Picard iteration for $T_1(x) = 0.2x$ and theoretical bounds

n	x_n	Banach	Kannan	Chatterjea	Reich	Hardy–Rogers	Ćirić
0	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000
1	0.2000000	0.2000000	0.3333333	0.2000000	0.2000000	0.2000000	0.2000000
2	0.0400000	0.0400000	0.1111111	0.0400000	0.0400000	0.0400000	0.0400000
3	0.0080000	0.0080000	0.0370370	0.0080000	0.0080000	0.0080000	0.0080000
4	0.0016000	0.0016000	0.0123457	0.0016000	0.0016000	0.0016000	0.0016000
5	0.0003200	0.0003200	0.0041152	0.0003200	0.0003200	0.0003200	0.0003200
6	0.0000640	0.0000640	0.0013717	0.0000640	0.0000640	0.0000640	0.0000640
7	0.0000128	0.0000128	0.0004572	0.0000128	0.0000128	0.0000128	0.0000128
8	0.00000256	0.00000256	0.0001524	0.00000256	0.00000256	0.00000256	0.00000256
9	0.000000512	0.000000512	0.0000508	0.000000512	0.000000512	0.000000512	0.000000512

Quadratic example $T_2(x) = 0.2x + 0.01x(1 - x)$:

For the nonlinear mapping T_2, the contractive constants must account for the larger derivative $T_2'(x) = 0.21 - 0.02x$, whose maximum on $[0, 1]$ is 0.21. Accordingly, we may take Banach constant $k = 0.21$; Kannan constant $k \approx 0.26$ (yielding contraction constant $c = \frac{k}{1-k} \approx 0.351$); Chatterjea constant $k \approx 0.17$ (giving $c \approx 0.205$); Reich parameters satisfying $\frac{\alpha}{1-\beta} \approx 0.21$; Hardy–Rogers parameters satisfying $\frac{a_1}{1-a_2} \approx 0.21$; and Ćirić constant $k = 0.21$.

Table 2. Picard iteration for $T_2(x) = 0.2x + 0.01x(1 - x)$ and theoretical bounds

n	x_n	Banach	Kannan	Chatterjea	Reich	Hardy–Rogers	Ćirić
0	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000
1	0.2000000	0.2100000	0.3510000	0.2050000	0.2100000	0.2100000	0.2100000
2	0.0416000	0.0441000	0.1232010	0.0420250	0.0441000	0.0441000	0.0441000
3	0.0087187	0.0092610	0.0432436	0.0086151	0.0092610	0.0092610	0.0092610
4	0.0018302	0.0019448	0.0151765	0.0017661	0.0019448	0.0019448	0.0019448
5	0.0003844	0.0004084	0.0053270	0.0003621	0.0004084	0.0004084	0.0004084
6	0.0000807	0.0000858	0.0018698	0.0000742	0.0000858	0.0000858	0.0000858
7	0.00001695	0.0000180	0.0006563	0.0000152	0.0000180	0.0000180	0.0000180
8	0.00000356	0.00000378	0.0002304	0.00000312	0.00000378	0.00000378	0.00000378
9	0.00000075	0.00000079	0.0000809	0.00000064	0.00000079	0.00000079	0.00000079

Tables 1 and 2 show something important. The contraction strength of mappings within the same contraction class is not necessarily uniform. This is illustrated by the given examples: the linear mapping has a contraction factor of exactly 0.2, whereas the quadratic (nonlinear) mapping has a contraction factor of approximately 0.2. Therefore, the numerical values of a mapping influence its contraction strength, even when compared to other mappings belonging to the same class.

Although the contraction constants of these two mappings differ, the Picard iterations used to construct sequences approximating their fixed points yield nearly identical convergence behavior. They are said to be nearly identical because both mappings reach a prescribed tolerance within the same number of steps (i.e., the same number of iterations).

Furthermore, when comparisons are made across contraction classes, the Kannan contractive condition consistently (in both given examples of mappings) yields weaker bounds than the other classes. In contrast, the remaining contraction classes behave uniformly and better. This can be observed at the 9th iteration, where Kannan produces values that are considerably farther from the fixed point compared to the other classes. Therefore, based on the results from both mappings, it can be concluded that the Kannan contractive condition is the slowest to achieve a given accuracy via Picard iteration.

4.5. Structural Comparison

While Proposition 4.1 establishes the sharp upper bounds for the contraction constants (i.e., the critical threshold values where the fixed point property may fail), the strict inequalities presented in Table 3 summarize the sufficient conditions that guarantee the existence and uniqueness of a fixed point for each classical contractive class. The table also highlights the distinctive contractive structure that characterizes each class.

Table 3. Structural properties of classical contractive mappings (with sharp constants)

Type	Sharp Constant	Contractive Structure	Fixed Point Uniqueness
Banach	$k < 1$	$d(x, y)$	Yes
Kannan	$k < \frac{1}{2}$	$d(x, Tx) + d(y, Ty)$	Yes
Chatterjea	$k < \frac{1}{2}$	$d(x, Ty) + d(y, Tx)$	Yes
Reich	$\alpha + \beta + \gamma < 1$	$\alpha d(x, y) + \beta d(x, Tx) + \gamma d(y, Ty)$	Yes
Hardy–Rogers	$\sum_{i=1}^{5} a_i < 1$	$a_1 d(x, y) + a_2 d(x, Tx) + a_3 d(y, Ty) + a_4 d(x, Ty) + a_5 d(y, Tx)$	Yes
Ćirić	$k < 1$	$k \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}$	Yes

The six classical classes of contractions offer essentially the same theoretical guarantees: existence and uniqueness of a fixed point in complete metric spaces, with no need to assume continuity. Nevertheless, their mathematical formulations are fundamentally different, both in algebraic form and in the values of their sharp constants.

All six contraction classes guarantee convergence and a fixed point, no exception. Their main difference lies in the formula they use. Banach works with the direct distance. Kannan and Chatterjea bring in distances to the image point. Reich and Hardy–Rogers combine several distances in a linear way. Ćirić, on the other hand, picks the maximum among them. The constant bounds given in the table are optimal; you cannot push them any higher. These structural differences give us a way to compare inclusion relationships and Picard iteration performance, especially looking at how tight the resulting convergence estimates are.

5. Conclusions

This paper compares six classical contraction classes from several angles: sharp constants, inclusion relationships, and Picard iteration performance. All six classes guarantee the existence and uniqueness of a fixed point under their respective strict contraction inequalities. We also establish a sufficient condition (specifically $k < \frac{1}{4}$) under which a Chatterjea contraction is also a Reich contraction. Moreover, we demonstrate that the sharp bound for each class marks the threshold beyond which the fixed point property may fail, thereby characterizing the maximal parameter range for universal guarantees.

Picard iteration is examined numerically for two representative mappings (one linear, one quadratic) that belong to all six classes. For these two mappings, the Kannan contraction requires more iterations to reach a given accuracy compared to the other classes, which is consistent with its theoretical a priori error bound $\frac{k}{1-2k}$ that suggests structurally slower convergence.

A limitation of this study is that only two examples were examined. While the qualitative ordering of convergence speeds (Banach fastest, Kannan slowest) is expected to remain robust for mappings with the same effective contraction factor, other choices of mappings may influence the quantitative differences in iteration counts. Future work should investigate broader families of mappings to further validate these observations.

Thus, this work offers a unified perspective blending structural analysis, sharp constants, and sequence behavior under Picard iteration.

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